Interacting Quantum Scalar Field Theory

相互作用量子标量场论

on a Causal Set

基于因果集

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Abstract

摘要

We introduce ϕ^4 interacting real scalar quantum field theory (QFT) on causal sets. We consider both the canonical framework of causal set free QFT, involving a Hilbert space and operators and so on, and the double path integral framework of causal set QFT outlaid by Sorkin. In both cases, we describe how to extend the formalism to include a ϕ^4 self-interaction, and, to make contact with the continuum, we contrast certain key expressions with their continuum counterparts. We develop a diagram-based algorithm, analogous to Feynman diagrams in the continuum, to compute the interacting 2-point function of our causal set QFT. Notably, causality is manifest in our diagrams in a manner not present in the usual Feynman diagrams of the continuum theory.

我们介绍因果集合上的 ϕ^4 相互作用实标量量子场论 (QFT)。我们同时考虑了包含希尔伯特空间、算符等内容的因果集合自由量子场论正则框架，以及由索金 (Sorkin) 提出的因果集合量子场论双路径积分框架。我们描述了如何在两种情形下扩展形式体系以纳入 ϕ^4 自相互作用，并且为了和连续理论建立联系，我们将若干关键表达式与其连续对应形式做了对比。我们开发了一种基于图的算法——类似于连续理论中的费曼图——来计算我们的因果集合量子场论的相互作用两点函数。值得注意的是，因果性在我们的图中是显然的，这一点是连续理论的常规费曼图所不具备的。

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Introduction

引言

For the sake of consistency of causal set quantum gravity with the standard model - the latter being described through the framework of quantum field theory (QFT) in a fixed background spacetime - it is imperative that we develop a theory of interacting quantum fields on causal sets and one that agrees, at least in some regime, with observation (That said, one could argue that disagreement at high energies could be interpreted as phenomenological evidence for causal set theory, though we will not explore this avenue here.). In a full theory of causal set quantum gravity, one imagines that spacetime itself would obey some quantum dynamics and that there would be some back-reaction between the spacetime and any quantum fields in the spacetime. Without such a complete description at hand, and in the hopes that causal set quantum gravity can, in some regime, reproduce the standard model, we permit ourselves to first address quantum fields on a fixed background causal set (When making comparisons to the continuum, it is useful to also consider many causal sets realized via Poisson sprinklings into some given continuum spacetime.).

为了让因果集量子引力与标准模型保持一致——后者是通过固定背景时空下的量子场论 (QFT) 框架描述的——我们必须发展出因果集上的相互作用量子场论，且该理论至少在某一能区要与观测相符 (话虽如此，也可以说，高能区的偏差可以被解释为因果集理论的现象学证据，但本文不探讨这一方向)。在完整的因果集量子引力理论中，一般认为时空本身遵循某种量子动力学，且时空与其中的任意量子场之间存在某种反作用。由于目前还没有这样完备的描述，且出于因果集量子引力能在某一能区重现标准模型的期望，我们允许自己先研究固定背景因果集上的量子场 (与连续统对比时，考虑多个通过泊松撒播进入给定连续时空得到的因果集会很有用)。

Our current description of quantum field theory (QFT) on a given fixed causal set is at a somewhat early stage of its development. Our lack of a description of tensors and/or spinors on causal sets limits us to only scalar fields, as opposed to fermionic or gauge fields. For this reason, we focus here on interacting real scalar quantum fields and, for concreteness, consider only the specific case of a self-interacting ϕ^4 theory.

我们目前对给定固定因果集上量子场论 (QFT) 的描述还处于发展初期。由于缺乏因果集上张量和/或旋量的描述方法，我们目前只能研究标量场，无法处理费米子场或规范场。因此，我们本文聚焦相互作用实标量量子场，为具体起见，仅考虑自相互作用 ϕ^4 理论这一特定情形。

Textbook approaches to interacting QFT in the continuum usually begin with the canonical description of the free theory, involving Hilbert spaces and operators and so on, and then move over to the path integral framework when interactions are introduced [1]. We will follow the same route here when introducing causal set interacting QFT. Before doing that, however, we recap some important aspects of the continuum theory in section ”Continuum Framework”. Much of our discussion surrounding the path integral approach to causal set QFT will actually be more concerned with the double path integral or Schwinger-Keldysh formalism, as this will be more appropriate for our purposes. The textbook approach to path integral QFT is not often expressed via a double path integral, however, and hence we first devote some time to deriving some useful expressions in the continuum that will provide the basis/motivation for the analogous causal set double path integral expressions.

连续统相互作用量子场论的教科书方法通常从自由理论的正则描述入手，涉及希尔伯特空间、算符等等，引入相互作用时再转用路径积分框架 [1]。我们在引入因果集相互作用量子场论时也遵循这一思路。但在此之前，我们会在“连续统框架”一节回顾连续统理论的一些重要内容。实际上，我们围绕因果集量子场论路径积分方法的大部分讨论都会更关注双路径积分，也就是施温格-凯尔迪什形式，因为它更适合我们的研究目的。而教科书的路径积分量子场论方法通常不会用双路径积分表述，因此我们先花时间在连续统下推导一些有用的表达式，为因果集对应的双路径积分表达式提供基础与动机。

The canonical [2] and double path integral [3] frameworks for free scalar QFT on causal sets are recapped in section ”Causal Set Free Scalar Field Theory”. Extensions to interacting fields have not yet been considered in detail in the literature. In [3], Sorkin notes that the double path integral approach is particularly advantageous in that it offers a clear route to introducing a self-interaction, namely, by adding the appropriate ϕ^4 term to the action [3]. This is precisely what we do in section ”Causal Set Interacting Scalar Field Theory”, and we find that, from the perspective of the canonical framework, one can interpret this modification to the action as a unitary transformation of the field operators. To finish, we focus on the 2-point function of the interacting theory in section ”Interacting 2-Point Function”, and in section ”The Analogue of Feynman Diagrams”, we derive some analogous Feynman rules for calculating it order by order in the interaction parameter. Our diagrams resemble those of the continuum theory, but with the added complication of two types of edges between vertices: directed and undirected. Our analogue Feynman rules are summarized for convenience in section ”Summary of Analogue Feynman Diagrams and Rules”.

我们会在“因果集自由标量场论”一节回顾因果集上自由标量量子场论的正则 [2] 和双路径积分 [3] 框架。目前文献中尚未详细讨论向相互作用场的推广。在文献 [3] 中，索金指出双路径积分方法的优势特别在于它提供了引入自相互作用的清晰路径，也就是在作用量中加入合适的 ϕ^4 项 [3]。这正是我们在“因果集相互作用标量场论”一节所做的工作，我们发现，从正则框架的视角来看，可以将这种对作用量的修改解释为场算符的幺正变换。最后，我们在“相互作用两点关联函数”一节聚焦相互作用理论的两点关联函数，又在“费曼图的类比”一节推导出了一些类比费曼规则，用以按相互作用参数的阶逐阶计算两点关联函数。我们的图和连续统理论中的费曼图类似，但额外多了顶点之间的两种边的复杂性: 有向边和无向边。为方便起见，我们在“类比费曼图与规则总结”一节总结了我们的类比费曼规则。

While there is not any published work on the approach presented here to interacting QFT on causal sets, we highlight Albertini’s thesis [4], which is the inspiration for much of what follows. We note, however, that our approach to the causal set analogue of Feynman diagrams differs from that in [4]. It is important to also highlight the work in [5], where interacting QFT on causal sets is discussed through the lens of deformation quantization. This method is somewhat more involved than the textbook approach considered here, and it would be of interest to determine if the two could be reconciled.

虽然目前没有公开发表的工作研究本文提出的因果集相互作用量子场论方法, 但我们要强调阿尔贝蒂尼的学位论文 [4]，本文后续大部分内容的灵感都来源于此。不过需要说明，我们对因果集费曼图类比的研究方法与文献 [4] 不同。我们还必须强调文献 [5] 中的工作，该工作从形变量子化的视角讨论了因果集相互作用量子场论。这种方法比本文采用的教科书方法更复杂，探究二者能否相容会很有意义。

Continuum Framework

连续统框架

Free Theory

自由理论

We first review some relevant concepts from real scalar QFT in Minkowski spacetime (Throughout we assume a mostly minus signature convention for the metric.) and derive some useful expressions in the double path integral framework.

我们首先回顾闵氏时空下实标量量子场论的相关概念 (全文我们采用度量的 Mostly Negative 符号约定)，并在双路径积分框架下推导出若干有用的表达式。

Consider first the free theory. An important quantity for our purposes, and indeed for much of QFT, is the time-ordered correlation function

首先考虑自由理论。对我们的研究而言，乃至对整个量子场论而言，一个十分重要的物理量就是时序关联函数

$$\langle 0 |\phi^{(H)}(x_2)\,\phi^{(H)}(x_1)| 0\rangle \tag{1}$$

where x_1 and x_2 are two spacetime points, with time coordinates ordered as $x_1^0 < x_2^0$. In what follows, spatial coordinates of a point x will be denoted as $\vec{x}$.$|0\rangle$ denotes the ground state of the theory, and $\phi^{(H)}(x) = U(0,x^0)^{\dagger}\phi^{(S)}(\vec{x})\,U(0,x^0)$ is the field operator (Technically, $\phi^{(H)}(x)$ is an operator-valued distribution that one must integrate against a test function to yield a well-defined operator on the Hilbert space.) at x in the Heisenberg picture. The Schrödinger picture field, $\phi^{(S)}(\vec{x})$, is only a function of the spatial coordinates. Here, $U(t,t')$ is the unitary operator that evolves the system from t to t' .

其中 x_1 和 x_2 是两个时空点，时间坐标满足顺序 $x_1^0 < x_2^0$ 。下文我们将点 x 的空间坐标记为 $\vec{x}$.$|0\rangle$, $\vec{x}$.$|0\rangle$ 表示该理论的基态，$\phi^{(H)}(x) = U(0,x^0)^{\dagger}\phi^{(S)}(\vec{x})\,U(0,x^0)$ 是海森堡绘景下 x 处的场算符 (严格来说，$\phi^{(H)}(x)$ 是算符值分布，必须对检验函数积分后才能得到希尔伯特空间上良定义的算符)。薛定谔绘景下的场 $\phi^{(S)}(\vec{x})$ 仅为空间坐标的函数。此处 $U(t,t')$ 是将系统从 t 演化到 t' 的幺正算符。

In Peskin and Schroeder, Chapter 9, p.283-4 [1], they derive the path integral representation of (1):

在 Peskin 与 Schroeder 所著教材第 9 章 283-284 页 [1] 中，他们推导出了式 (1) 的路径积分表示:

$$\langle 0 |\phi^{(H)}(x_2)\,\phi^{(H)}(x_1)| 0\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathcal{D}\xi\,\xi(x_1)\,\xi(x_2)\,e^{iS[\xi]}}{\int \mathcal{D}\xi\, e^{iS[\xi]}}, \tag{2}$$

where $S[\xi] = \int_{-T}^{T} d^d x \mathcal{L}$ is the action from time $-T$ to T , and the integrals are over all classical spacetime field configurations $\xi(x) \equiv \xi(x^0,\vec{x})$ with the boundary conditions $\xi(\pm T,\vec{x}) = \zeta_{\pm}(\vec{x})$, where $\zeta_{\pm}(\vec{x})$ are some functions of the spatial coordinates only. The specific forms of the boundary functions $\zeta_{\pm}$ are actually irrelevant in the limit $T \to \infty(1-i\varepsilon)$ (provided the support of the ground state wavefunctional includes these functions).

其中 $S[\xi] = \int_{-T}^{T} d^d x \mathcal{L}$ 是时间从 $-T$ 到 T 的作用量，积分遍及所有满足边界条件 $\xi(\pm T,\vec{x}) = \zeta_{\pm}(\vec{x})$ 的经典时空场构型 $\xi(x) \equiv \xi(x^0,\vec{x})$，$\zeta_{\pm}(\vec{x})$ 是仅依赖空间坐标的函数。当取极限 $T \to \infty(1-i\varepsilon)$ 时，边界函数 $\zeta_{\pm}$ 的具体形式其实并不相关 (只要基态波泛函的支集包含这些函数)。

Remark. In the interacting theory, the act of taking $T \to \infty(1-i\varepsilon)$ also ensures that one recovers the correlation function in the ground state of the interacting theory. In the causal set case below, we only consider finite causal sets, and hence we do not have the luxury of taking $T \to \infty$, let alone in a slightly imaginary direction. It is not clear, then, how one can recover the ground state of the interacting theory in the causal set case. We comment briefly on this at the end of section "Interacting 2-Point Function" and for now content ourselves with the 2-point function in an arbitrary state $|\Psi\rangle$, i.e., $\langle \Psi |\phi^{(H)}(x_2)\,\phi^{(H)}(x_1)| \Psi\rangle$. |

注记: 在相互作用理论中，取 $T \to \infty(1-i\varepsilon)$ 还能保证我们得到相互作用理论基态下的关联函数。下文讨论因果集情形时,我们仅考虑有限因果集,因此无法取 $T \to \infty$ 极限,更不用说沿微小虚方向取极限了。因此目前尚不清楚在因果集情形下如何得到相互作用理论的基态。我们会在“相互作用两点关联函数”一节末尾对此简要讨论，目前我们满足于得到任意态 $|\Psi\rangle$, i.e., $\langle\Psi|\phi^{(H)}(x_2)\phi^{(H)}(x_1)|\Psi\rangle$. | 下的两点关联函数。

In what follows, the symbol ξ will be used exclusively for functions of both space and time coordinates. For brevity, we also drop the $\vec{x}$ argument in $\xi(t,\vec{x})$ and simply write $\xi(t)$ when referring to the spatial function one obtains by evaluating ξ at time t . Finally, the symbol ζ will be used exclusively for functions just of spatial coordinates.

下文将把符号 ξ 专门用于表示同时依赖空间坐标和时间坐标的函数。为简洁起见，我们在指代将 ξ 在时刻 t 处求值得到的空间函数时，会省略 $\xi(t,\vec{x})$ 中的 $\vec{x}$ 自变量，直接记作 $\xi(t)$ 。最后，符号 ζ 将专门用于表示仅依赖空间坐标的函数。

It will be convenient for our purposes to express $\langle\Psi|\phi^{(H)}(x_2)\phi^{(H)}(x_1)|\Psi\rangle$ not as a single path integral as in (2), but as a double path integral. To do this, we first express the fields in the Schrödinger picture. We have

对我们的研究而言，将 $\langle\Psi|\phi^{(H)}(x_2)\phi^{(H)}(x_1)|\Psi\rangle$ 表示为双路径积分而非式 (2) 中的单路径积分会更方便。为此，我们首先将场用薛定谔绘景写出，可得

$$\langle\Psi|\phi^{(H)}(x_2)\phi^{(H)}(x_1)|\Psi\rangle$$

$$= \left\langle\Psi\left|U(0,T)^{\dagger}U\left(x_2^0,T\right)\phi^{(S)}\left(\vec{x}_2\right)U\left(x_1^0,x_2^0\right)\phi^{(S)}\left(\vec{x}_1\right)U\left(0,x_1^0\right)\right|\Psi\right\rangle, \tag{3}$$

for some arbitrary $T > x_{1,2}^0$. We next express the bra and kets in (3) as

对任意 $T > x_{1,2}^0$ 成立。接下来我们将式 (3) 中的左矢和右矢表示为

$$\langle\Psi\Big| = \int \mathcal{D}\zeta'\Psi(\zeta')^{*}\langle\zeta'|, |\Psi\rangle = \int \mathcal{D}\zeta\Psi(\zeta)\Big|\zeta\rangle, \tag{4}$$

where the integrals are over all spatial configurations $\zeta \equiv \zeta(\vec{x})$ and Ψ is the wavefunctional that returns a complex number for every spatial configuration ζ .

其中积分遍历所有空间构型 $\zeta \equiv \zeta(\vec{x})$ ，Ψ 是为每个空间构型 ζ 给出一个复数值的波泛函。

In terms of this field configuration basis, we similarly express the Schrödinger field operators in (3) as

利用该场构型基，我们可以类似地将式 (3) 中的薛定谔场算符表示为

$$\phi^{(S)}(\vec{x}_2) = \int \mathcal{D}\zeta_2\zeta_2(\vec{x}_2)|\zeta_2\rangle\langle\zeta_2|,\ \phi^{(S)}(\vec{x}_1) = \int \mathcal{D}\zeta_1\zeta_1(\vec{x}_1)|\zeta_1\rangle\langle\zeta_1|. \tag{5}$$

Finally, we insert the identity,

最后，我们插入恒等式

$$\mathbb{1} = \int \mathcal{D}\zeta_T \left|\zeta_T\right\rangle \left\langle\zeta_T\right| \tag{6}$$

in between $U(0,T)^\dagger$ and $U\left(x_2^0, T\right)$ in (3). We then have

到式 (3) 中的 $U(0,T)^\dagger$ 和 $U\left(x_2^0, T\right)$ 之间，即可得到

$$\begin{aligned}
\left\langle\Psi\left|\phi^{(H)}\left(x_2\right)\phi^{(H)}\left(x_1\right)\right|\Psi\right\rangle &= \int \mathcal{D}\zeta'\Psi(\zeta')^* \left\langle\zeta'\left|U(0,T)^\dagger \int \mathcal{D}\zeta_T\right|\zeta_T\right\rangle\left\langle\zeta_T\right| U\left(x_2^0, T\right) \\
&\times \int \mathcal{D}\zeta_2\zeta_2\left(\vec{x}_2\right)\left|\zeta_2\right\rangle\left\langle\zeta_2\left|U\left(x_1^0, x_2^0\right)\int \mathcal{D}\zeta_1\zeta_1\left(\vec{x}_1\right)\right|\zeta_1\right\rangle\left\langle\zeta_1\right| \\
&\times U\left(0, x_1^0\right)\int \mathcal{D}\zeta\Psi\left(\zeta\right)\left|\zeta\right\rangle \\
&= \int \mathcal{D}\zeta'\mathcal{D}\zeta_T\mathcal{D}\zeta_2\mathcal{D}\zeta_1\mathcal{D}\zeta\Psi(\zeta')^*\Psi\left(\zeta\right)\left\langle\zeta'\left|U(0,T)^\dagger\right|\zeta_T\right\rangle \\
&\times\left\langle\zeta_T\left|U\left(x_2^0,T\right)\right|\zeta_2\right\rangle\left\langle\zeta_2\left|U\left(x_1^0,x_2^0\right)\right|\zeta_1\right\rangle\left\langle\zeta_1\left|U\left(x_1^0,0\right)\right|\zeta\right\rangle \\
&\times\zeta_1\left(\vec{x}_1\right)\zeta_2\left(\vec{x}_2\right)
\end{aligned} \tag{7}$$

Next, we can rewrite each of the unitary transition amplitudes using the following identity [1]:

接下来，我们可以利用下述恒等式重写每个幺正跃迁振幅 [1]:

$$\left\langle\zeta'\left|U\left(t,t'\right)\right|\zeta\right\rangle = \int_{\xi(t')=\zeta'} \mathcal{D}\xi e^{iS[\xi]}, \tag{8}$$

where the integral is over all spacetime field configurations $\xi \equiv \xi\left(x\right)$ between the times t and t' , with boundary values fixed on those time slices to the spatial functions ζ and ζ' , respectively. Note that $S\left[\xi\right] = \int_t^{t'} d^d x\mathcal{L}$ denotes the action from t to t' in this case.

其中积分遍历时间 t 和 t' 之间的所有时空场构型 $\xi \equiv \xi\left(x\right)$ ，这两个时间切片上的边界值分别固定为空间函数 ζ 和 ζ' 。注意，此处 $S\left[\xi\right] = \int_t^{t'} d^d x\mathcal{L}$ 表示从 t 到 t' 的作用量。

Using the identity (8), the rhs of (7) then becomes

利用恒等式 (8)，式 (7) 的右端可变为

$$\int \mathcal{D}\zeta'\mathcal{D}\zeta_T\mathcal{D}\zeta_2\mathcal{D}\zeta_1\mathcal{D}\zeta\int_{\substack{\bar{\xi}\left(T\right)=\zeta_T \\ \bar{\xi}\left(0\right)=\zeta_T}}\mathcal{D}\bar{\xi}\begin{cases}\xi_{\xi(T)=\zeta_T}\mathcal{D}\bar{\xi}e^{i\left(S[\xi]-S\left[\bar{\xi}\right]\right)}\Psi(\zeta')^*\Psi\left(\zeta\right)\zeta_2\left(\vec{x}_2\right)\zeta_1\left(\vec{x}_1\right), \\ \xi_{\xi(1)=\zeta_1} = \zeta_1 \\ \xi_{\xi(1)=\zeta} = \zeta_1\end{cases}$$

(9)where the action is now from 0 to T . As the spacetime field configuration ξ is pinned to ζ_1 at time x_1^0 , and to ζ_2 at time x_2^0 , we can replace $\zeta_2\left(\vec{x}_2\right)\zeta_1\left(\vec{x}_1\right)$ by $\xi\left(x_2\right)\xi\left(x_1\right)$ in the integrand. The six integrals in (9) amount to a double path integral over all pairs of spacetime field configurations $\bar{\xi}$ and ξ , but with the restriction that the two configurations agree at time T . This restriction can be encoded via the delta function $\delta\left(\xi\left(T\right)-\bar{\xi}\left(T\right)\right)$. We now have our final expression:

其中作用量现在是从 0 到 T 的作用量。由于时空场构型 ξ 在时刻 x_1^0 被固定为 ζ_1 ， 在时刻 x_2^0 被固定为 ζ_2 ， 我们可以将被积函数中的 $\zeta_2\left(\vec{x}_2\right)\zeta_1\left(\vec{x}_1\right)$ 替换为 $\xi\left(x_2\right)\xi\left(x_1\right)$ 。式 (9) 中的六个积分等价于对所有时空场构型对 $\bar{\xi}$ 和 ξ 的双路径积分，但额外要求两个构型在时刻 T 相等。该约束可以通过 δ 函数 $\delta\left(\xi\left(T\right)-\bar{\xi}\left(T\right)\right)$ 来表示。我们由此得到最终表达式:

$$
\begin{aligned}
&\left\langle\Psi\left|\phi^{(H)}\left(x_2\right)\phi^{(H)}\left(x_1\right)\right|\Psi\right\rangle \\
&=\int\mathcal{D}\bar{\xi}\mathcal{D}\xi e^{i\left(S[\xi]-S\left[\bar{\xi}\right]\right)}\delta\left(\xi\left(T\right)-\bar{\xi}\left(T\right)\right)\Psi\left(\bar{\xi}\left(0\right)\right)^*\Psi\left(\xi\left(0\right)\right)\xi\left(x_2\right)\xi\left(x_1\right). \qquad (10)
\end{aligned}
$$

We can rewrite this expression as

我们可以将该表达式重写为

$$
\left\langle\Psi\left|\phi^{(H)}\left(x_2\right)\phi^{(H)}\left(x_1\right)\right|\Psi\right\rangle=\int\mathcal{D}\bar{\xi}\mathcal{D}\xi D\left(\xi,\bar{\xi}\right)\xi\left(x_2\right)\xi\left(x_1\right), \qquad (11)
$$

where we have introduced the decoherence functional

其中我们引入了退相干泛函

$$
D\left(\xi,\bar{\xi}\right)=e^{i\left(S[\xi]-S\left[\bar{\xi}\right]\right)}\delta\left(\xi\left(T\right)-\bar{\xi}\left(T\right)\right)\Psi\left(\bar{\xi}\left(0\right)\right)^*\Psi\left(\xi\left(0\right)\right). \qquad (12)
$$

Interacting Theory

相互作用理论

To move to the interacting theory, one modifies the double path integral expression by adding an interaction term to the action, $S\left[\xi\right]\mapsto S\left[\xi;\lambda\right]=S\left[\xi\right]+S_{\text{int}}\left[\xi;\lambda\right]$, where $S\left[\xi\right]$ is the action of the free theory and λ is the coupling or interaction parameter. The interacting decoherence functional is then

为了得到相互作用理论，我们通过在作用量中添加相互作用项修改了双路径积分表达式，即 $S\left[\xi\right]\mapsto S\left[\xi;\lambda\right]=S\left[\xi\right]+S_{\text{int}}\left[\xi;\lambda\right]$ ， 其中 $S\left[\xi\right]$ 是自由理论的作用量，λ 是耦合或相互作用参数。由此得到相互作用退相干泛函为

$$
\begin{aligned}
D\left(\xi,\bar{\xi};\lambda\right)&=e^{i\left(S[\xi;\lambda]-S\left[\bar{\xi};\lambda\right]\right)}\delta\left(\xi\left(T\right)-\bar{\xi}\left(T\right)\right)\Psi\left(\bar{\xi}\left(0\right)\right)^*\Psi\left(\xi\left(0\right)\right) \\
&=D\left(\xi,\bar{\xi}\right)e^{i\left(S_{\text{int}}[\xi;\lambda]-S_{\text{int}}\left[\bar{\xi};\lambda\right]\right)}. \qquad (13)
\end{aligned}
$$

Following the logic of section "Free Theory" in reverse, if we compute the integral in (11) with $D\left(\xi,\overline{\xi}\right)$ replaced by $D\left(\xi,\overline{\xi};\lambda\right)$, i.e., $\int\mathcal{D}\overline{\xi}\mathcal{D}\xi D\left(\xi,\overline{\xi};\lambda\right)\xi\left(x_2\right)\xi\left(x_1\right)$, we find $\langle\Psi\left|\phi^{(H)}\left(x_2\right)\phi^{(H)}\left(x_1\right)\right|\Psi\rangle$ as we did in the free case. It is important to note, however, that even though the free and interacting double path integrals amount to the same expression in terms of the Heisenberg picture fields, i.e., $\langle\Psi\left|\phi^{(H)}\left(x_2\right)\phi^{(H)}\left(x_1\right)\right|\Psi\rangle$, the 2-point functions are different due to differences in the Heisenberg picture fields themselves. In the free theory, one finds the Heisenberg fields by evolving the Schrödinger picture fields with the free Hamiltonian, H_0 , as $\phi^{(H)}(x) = e^{iH_0x^0}\phi^{(S)}\left(\vec{x}\right)e^{-iH_0x^0}$, whereas in the interacting theory, they are evolved with the full Hamiltonian, $H(t) = H_0 + H_{\text{int}}(t)$, as $\phi^{(H)}(x) = T\left\{e^{i\int_0^{x^0}dtH(t)}\phi^{(S)}\left(\vec{x}\right)T\left\{e^{-i\int_0^{x^0}dtH(t)}\right\}\right\}$. In making comparisons between the causal set and continuum field theories below, it will be more convenient to focus on the interaction picture fields in the interacting theory - those that are evolved with the free Hamiltonian as $\phi^{(I)}(x) = e^{iH_0x^0}\phi^{(S)}\left(\vec{x}\right)e^{-iH_0x^0}$. It should be clear from their definition that they are the same as the Heisenberg fields in the free theory. Henceforth, any use of $\phi(x)$ without a superscript should be understood as an interaction picture field, unless otherwise stated.

遵循“自由理论”小节的反向推导逻辑，如果我们将 (11) 式中 $D\left(\xi,\overline{\xi}\right)$ 替换为 $D\left(\xi,\overline{\xi};\lambda\right)$ ，也就是 $\int\mathcal{D}\overline{\xi}\mathcal{D}\xi D\left(\xi,\overline{\xi};\lambda\right)\xi\left(x_2\right)\xi\left(x_1\right)$ ，我们可以像自由情况一样得到 $\langle\Psi\left|\phi^{(H)}\left(x_2\right)\phi^{(H)}\left(x_1\right)\right|\Psi\rangle$ 。但需要注意的是，即使自由理论和相互作用理论的双路径积分在海森堡绘景场下得到了相同的表达式，即 $\langle\Psi\left|\phi^{(H)}\left(x_2\right)\phi^{(H)}\left(x_1\right)\right|\Psi\rangle$ ，由于海森堡绘景场本身存在差异，两点函数并不相同。在自由理论中，海森堡场通过自由哈密顿量 H_0 演化薛定谔绘景场得到，即 $\phi^{(H)}(x) = e^{iH_0x^0}\phi^{(S)}\left(\vec{x}\right)e^{-iH_0x^0}$ ；而在相互作用理论中，海森堡场由总哈密顿量 $H(t) = H_0 + H_{\text{int}}(t)$ 演化得到，即 $\phi^{(H)}(x) = T\left\{e^{i\int_0^{x^0}dtH(t)}\phi^{(S)}\left(\vec{x}\right)T\left\{e^{-i\int_0^{x^0}dtH(t)}\right\}\right\}$ 。在下文比较因果集和连续统场论时，我们关注相互作用理论中的相互作用绘景场会更方便——这类场和自由理论一样由自由哈密顿量演化，即 $\phi^{(I)}(x) = e^{iH_0x^0}\phi^{(S)}\left(\vec{x}\right)e^{-iH_0x^0}$ 。根据其定义可知，它们和自由理论中的海森堡场完全等价。除非另有说明，此后所有不带上标的 $\phi(x)$ 都应理解为相互作用绘景场。

In terms of the interaction picture fields, one can verify that

利用相互作用绘景场，可以验证得到

$$\langle\Psi\left|\phi^{(H)}\left(x_2\right)\phi^{(H)}\left(x_1\right)\right|\Psi\rangle = \langle\Psi\left|V(0,T)^{\dagger}V\left(x_2^0,T\right)\phi^{(I)}\left(x_2\right)V\left(x_1^0,x_2^0\right)\phi^{(I)}\left(x_1\right)V\left(0,x_1^0\right)\right|\Psi\rangle,$$

(14)

where

其中

$$V\left(t,t'\right) = T\left\{\exp\left(-i\int_t^{t'}dt''H_{\text{int}}^{(I)}\left(t''\right)\right)\right\}, \tag{15}$$

and $H_{\text{int}}^{(I)}(t) = e^{iH_0t}H_{\text{int}}(t)e^{-iH_0t}$ is the interacting part of the full Hamiltonian expressed in the interaction picture. For a self-interacting theory, say ϕ^4 , this would take the form of an integral over some spatial surface $\sum$ of constant time t :

且 $H_{\text{int}}^{(I)}(t) = e^{iH_0 t} H_{\text{int}}(t) e^{-iH_0 t}$ 是相互作用绘景下总哈密顿量的相互作用部分。对于自相互作用理论，例如 ϕ^4，其形式为在等时 t 的某个空间曲面 $\sum$ 上的积分:

$$H_{\text{int}}^{(I)}(t) = -\int_{\Sigma} d^{d-1}\vec{x} \frac{\lambda}{4!} \phi(x)^4, \tag{16}$$

where $x = (t, \vec{x})$, and where $\phi(x) \equiv \phi^{(I)}(x)$ is in the interaction picture. Here, λ is the coupling parameter.

其中满足 $x = (t, \vec{x})$，且 $\phi(x) \equiv \phi^{(I)}(x)$ 处于相互作用绘景。此处，λ 为耦合参数。

Before moving on, we note that to recover higher time-ordered n-point functions, e.g., $\langle \Psi | T \{ \phi(x_n) \dots \phi(x_1) \} | \Psi \rangle$, from the double path integral framework, one simply inserts $\xi(x_i)$ in the double path integrand for each $i = 1, \dots, n$.

在继续推导之前，我们注意到，若要从双路径积分框架中恢复高阶时序 n 点函数（例如 $\langle \Psi | T \{ \phi(x_n) \dots \phi(x_1) \} | \Psi \rangle$），只需对每个 $i = 1, \dots, n$ 在双路径被积函数中插入 $\xi(x_i)$ 即可。

There is an asymmetry here in how we are using the two path integrals over ξ and $\bar{\xi}$, since we are not inserting any field variables like $\bar{\xi}(x_i)$ into the integrand. This is a choice, and we could have reformulated the above using insertions of $\bar{\xi}(x_i)$ variables only. In a moment, we will see that, if we ask for more than n-point correlation functions - specifically for probabilities of measurement outcomes - then both ξ and $\bar{\xi}$ will be utilized symmetrically.

我们对 ξ 和 $\bar{\xi}$ 的两个路径积分的使用存在不对称性, 因为我们没有向被积函数中插入任何类似 $\bar{\xi}(x_i)$ 的场变量。这是一个选择，我们本可以仅通过插入 $\bar{\xi}(x_i)$ 变量来重新表述上述内容。稍后我们会看到，如果我们研究的不只是 n 点关联函数——具体来说是测量结果的概率——那么 ξ 和 $\bar{\xi}$ 将得到对称的运用。

Expressing the Decoherence Functional in the Canonical Framework

在正则框架下表示退相干泛函

To motivate our causal set expressions in sections "Histories Framework" and "Introducing an Interaction", it will be helpful to re-express the decoherence functionals in (12) and (13) in terms of the quantities of the canonical framework. This will be a somewhat formal discussion, as certain steps in this process are not well defined. In the causal set case, however, such steps are all well defined.

为了给我们在“历史框架”和“引入相互作用”章节中的因果集表达式提供动机，我们将把式 (12) 和 (13) 中的退相干泛函重新用正则框架的量表示，这会很有帮助。本次讨论在一定程度上是形式化的，因为该过程中的部分步骤并未得到良好定义。但在因果集的情况下，这些步骤都是良定义的。

Let us say we want to calculate the probability (or more accurately the probability density) that a measurement of $\phi(x_1)$ at x_1 (so at time x_1^0 and position $\vec{x}_1$) gives a value of $\eta_1 \in \mathbb{R}$ and that a measurement of

$\phi(x_2)$ gives a value of $\eta_2 \in \mathbb{R}$.

假设我们想要计算下述过程的概率 (更准确地说是概率密度): 在 x_1 测量 $\phi(x_1)$ (即在时间 x_1^0 、位置 $\vec{x}_1$ 处测量) 得到结果 $\eta_1 \in \mathbb{R}$ ，并且对 $\phi(x_2)$ 的测量得到结果 $\eta_2 \in \mathbb{R}$ 。

To determine this probability, we first evolve our initial state $|\Psi\rangle$ to time $x_1^0 : |\Psi\rangle \mapsto U\left(0, x_1^0\right)|\Psi\rangle$. We then project with the (formal) Schrödinger picture projector associated with the observation of $\eta_1 \; : \; U\left(0, x_1^0\right)\left|\Psi\right\rangle \mapsto P_1^{(S)} U\left(0, x_1^0\right)\left|\Psi\right\rangle$, where $P_1^{(S)} \; := \; \int_{\zeta_1 = \eta_1} \mathcal{D}\zeta_1 \left|\zeta_1\right\rangle\left\langle\zeta_1\right|$ and where the integral is over all spatial field configurations ζ_1 such that $\zeta_1\left(\vec{x}_1\right) = \eta_1$. We then evolve to x_2^0 with $U\left(x_1^0, x_2^0\right)$ and finally project with $P_2^{(S)}$, the associated projector for our observation of η_2 . Our desired probability is then the modulus squared of this final state:

为了得到这个概率，我们首先将初态 $|\Psi\rangle$ 演化到时刻 $x_1^0 : |\Psi\rangle \mapsto U\left(0, x_1^0\right)|\Psi\rangle$ 。然后我们用与观测 $\eta_1 \; : \; U\left(0, x_1^0\right)\left|\Psi\right\rangle \mapsto P_1^{(S)} U\left(0, x_1^0\right)\left|\Psi\right\rangle$ 相关的 (形式化的) 薛定谔绘景投影算子做投影，其中 $P_1^{(S)} := \int_{\zeta_1 = \eta_1} \mathcal{D}\zeta_1 \left|\zeta_1\right\rangle\left\langle\zeta_1\right|$ ，且积分遍历所有满足 $\zeta_1\left(\vec{x}_1\right) = \eta_1$ 的空间场构型 ζ_1 。之后我们用 $U\left(x_1^0, x_2^0\right)$ 将态演化到 x_2^0 ，最后用对应观测 η_2 的投影算子 $P_2^{(S)}$ 再次做投影。我们想要的概率就是这个终态的模平方:

$$\left\| P_2^{(S)} U\left(x_1^0, x_2^0\right) P_1^{(S)} U\left(0, x_1^0\right)|\Psi\rangle \right\|^2$$

$$= \left\| P_2^{(H)} P_1^{(H)} |\Psi\rangle \right\|^2 = \left\| P_2^{(I)} V\left(x_1^0, x_2^0\right) P_1^{(I)} V\left(0, x_1^0\right)|\Psi\rangle \right\|^2, \tag{17}$$

where we have also included the Heisenberg and interaction picture expressions for convenience. Focusing on the interaction picture expression (as this will be directly comparable when we consider causal sets below), we can then derive the corresponding double path integral expression as follows:

此处我们也为方便附上了海森堡绘景和相互作用绘景下的表达式。聚焦于相互作用绘景的表达式 (因为我们接下来讨论因果集时可以直接对比), 我们可以推导得到对应的双重路径积分表达式如下:

$$\left\| P_2^{(I)} V\left(x_1^0, x_2^0\right) P_1^{(I)} V\left(0, x_1^0\right)|\Psi\rangle \right\|^2$$

$$= \left(\left\langle \Psi \left| V(0, x_1^0)^\dagger P_1^{(I)} V(x_1^0, x_2^0)^\dagger P_2^{(I)} \right) \left(P_2^{(I)} V(x_1^0, x_2^0) P_1^{(I)} V(0, x_1^0) \right| \Psi \right\rangle\right)$$

$$= \left\langle \Psi \left| V(0, x_1^0)^\dagger P_1^{(I)} V(x_1^0, x_2^0)^\dagger P_2^{(I)} V(x_2^0, T)^\dagger V\left(x_2^0, T\right) P_2^{(I)} V\left(x_1^0, x_2^0\right) P_1^{(I)} V\left(0, x_1^0\right) \right| \Psi \right\rangle$$

$$= \int_{\substack{\bar{\xi}(x_1) = \eta_1 \\ \bar{\xi}(x_2) = \eta_2}} \mathcal{D}\bar{\xi} \int_{\substack{\xi(x_1) = \eta_1 \\ \xi(x_2) = \eta_2}} \mathcal{D}\xi D\left(\xi, \bar{\xi}\lambda\right), \tag{18}$$

where in the last line the two path integrals are over all field configurations ξ and $\bar{\xi}$ which evaluate to η_1 at x_1 and η_2 at x_2 , respectively. Here, the two path integrals have arisen, in some sense, from the two brackets in the second line of (18).

其中最后一行的两个路径积分分别遍历所有场构型 ξ 和 $\bar{\xi}$ ，它们分别在 x_1 处取值为 η_1 、在 x_2 处取值为 η_2 。从某种意义上说，这两个路径积分来自式 (18) 第二行的两个括号。

We can encode the restrictions on the path integrals via the delta functions $\delta\left(\xi\left(x_1\right)-\eta_1\right)\delta\left(\xi\left(x_2\right)-\eta_2\right)$ and similarly for $\bar{\xi}$. Following the logic of section "Free Theory", this allows us to reverse engineer the corresponding expression in the canonical framework:

> 我们可以通过 δ 函数 $\delta\left(\xi\left(x_1\right)-\eta_1\right)\delta\left(\xi\left(x_2\right)-\eta_2\right)$ 来编码对路径积分的约束，对 $\bar{\xi}$ 同理。遵循“自由理论”章节的逻辑，我们可以由此反推出正则框架下的对应表达式:

$$\int_{\substack{\bar{\xi}\left(x_1\right)=\eta_1\\ \bar{\xi}\left(x_2\right)=\eta_2}}\mathcal{D}\bar{\xi}\int_{\substack{\xi\left(x_1\right)=\eta_1\\ \bar{\xi}\left(x_2\right)=\eta_2}}\mathcal{D}\xi D\left(\xi,\bar{\xi};\lambda\right)$$

$$=\int\mathcal{D}\bar{\xi}\mathcal{D}\xi D\left(\xi,\bar{\xi};\lambda\right)\delta\left(\bar{\xi}\left(x_1\right)-\eta_1\right)\delta\left(\bar{\xi}\left(x_2\right)-\eta_2\right)\delta\left(\xi\left(x_2\right)-\eta_2\right)\delta\left(\xi\left(x_1\right)-\eta_1\right)$$

$$=\left\langle\Psi\middle|V\left(0,x_1^0\right)^{\dagger}\delta\left(\phi\left(x_1\right)-\eta_1\right)V\left(x_1^0,x_2^0\right)^{\dagger}\delta\left(\phi\left(x_2\right)-\eta_2\right)V\left(x_2^0,T\right)^{\dagger}\right.$$

$$\times V\left(x_2^0,T\right)\delta\left(\phi\left(x_2\right)-\eta_2\right)V\left(x_1^0,x_2^0\right)\delta\left(\phi\left(x_1\right)-\eta_1\right)V\left(0,x_1^0\right)|\Psi\rangle, \tag{19}$$

where we recall that by $\phi(x)$ we mean the interaction picture fields evolved with the free Hamiltonian. It is important to note here the resulting order of delta functions in the last line of (19), as these delta functions cannot be reordered if the fields $\phi\left(x_1\right)$ and $\phi\left(x_2\right)$ do not commute. From left to right, we first have the reverse-time-ordered product of $\delta\left(\phi\left(x_1\right)-\eta_1\right)$ and $\delta\left(\phi\left(x_2\right)-\eta_2\right)$, with the appropriate interaction picture unitaries, $V(\cdot,\cdot)^{\dagger}$, inserted in between. After this, we then have the time-ordered product of $\delta\left(\phi\left(x_1\right)-\eta_1\right)$ and $\delta\left(\phi\left(x_2\right)-\eta_2\right)$, again with appropriate insertions of $V(\cdot,\cdot)$.

> 此处我们回顾，$\phi(x)$ 指的是由自由哈密顿量演化得到的相互作用绘景场。需要特别注意 (19) 最后一行中 δ 函数的最终顺序: 若场 $\phi\left(x_1\right)$ 与 $\phi\left(x_2\right)$ 不对易，这些 δ 函数无法重排序。从左到右，首先是 $\delta\left(\phi\left(x_1\right)-\eta_1\right)$ 与 $\delta\left(\phi\left(x_2\right)-\eta_2\right)$ 的逆时间序乘积，中间插入了合适的相互作用绘景幺正算符 $V(\cdot,\cdot)^{\dagger}$ 。在此之后，是 $\delta\left(\phi\left(x_1\right)-\eta_1\right)$ 与 $\delta\left(\phi\left(x_2\right)-\eta_2\right)$ 的时间序乘积，同样插入了合适的 $V(\cdot,\cdot)$ 。

This precise ordering of these delta functions, and the precise insertions of $V(\cdot,\cdot)^{\dagger}$ and $V(\cdot,\cdot)$ in between them, generalizes to any finite number of spacetime points $x_1,\ldots,x_N$. We can further consider the case where the field configurations ξ and $\bar{\xi}$ are pinned to different values at x_i , say $\xi\left(x_i\right)=\eta_i$ and $\bar{\xi}_i=\bar{\eta}_i$.

> 这种 δ 函数的精确排序，以及在它们之间插入 $V(\cdot,\cdot)^{\dagger}$ 和 $V(\cdot,\cdot)$ 的精确方式，可以推广到任意有限个时空点 $x_1,\ldots,x_N$ 的情况。我们还可以进一步考虑场构型 ξ 和 $\bar{\xi}$ 在 x_i 处被固定为不同值的情况，例如固定为 $\xi\left(x_i\right)=\eta_i$ 和 $\bar{\xi}_i=\bar{\eta}_i$ 。

In fact, it will be useful to write down the explicit expression for a lattice of spacetime points, as this will help motivate some of our causal set expressions in sections "Histories Framework" and "Introducing an Interaction". Consider, then, a regular square lattice of points, Λ , embedded in a d -dimensional spacetime M , truncated in time from $t=0$ to $t=T$, and whose constant time surfaces, $\sum$, are of compact topology. Consider a regular lattice in M with finite lattice spacing and only a finite number of points, i.e., $|\Lambda|<\infty$. This ensures we do not have to deal with any troublesome infinities in what follows. For convenience, let the set of time coordinates of the points in our lattice be $\mathcal{T}=\{\Delta t,2\Delta t,\ldots,T-\Delta t\}$, where Δt is the lattice spacing in time. Let us also denote the set of all spatial coordinates of our lattice points as $\mathcal{X}$. We assume our lattice can then be expressed as the Cartesian product $\Lambda = \mathcal{T} \times \mathcal{X}$.

事实上，写出时空点格点的显式表达式是很有帮助的，这将为我们在“历史框架”和“引入相互作用”两节中推导因果集表达式提供动机。现在考虑一个嵌入在 d 维时空 M 中的规则正方形点格 Λ ，它在时间方向上从 $t=0$ 截断到 $t=T$ ，其等时面 $\sum$ 具有紧致拓扑。我们考察的规则格点 M 具有有限格距且仅含有限个点，即 $|\Lambda| < \infty$ 。这保证我们在后续推导中不需要处理棘手的无穷大。为方便起见，设格点中所有点的时间坐标集合为 $\mathcal{T} = \{\Delta t, 2\Delta t, \ldots, T-\Delta t\}$ ，其中 Δt 是时间方向的格距。再将格点所有空间坐标的集合记为 $\mathcal{X}$ 。我们的格点就可以表示为笛卡尔积 $\Lambda = \mathcal{T} \times \mathcal{X}$ 。

Now, consider two spacetime field configurations $\eta(x)$ and $\overline{\eta}(x)$ which we will pin ξ and $\overline{\xi}$ to, respectively, for each point in our lattice. We then have

现在考虑两个时空场构型 $\eta(x)$ 和 $\overline{\eta}(x)$ ，对于格点中的每个点，我们分别将 ξ 和 $\overline{\xi}$ 固定在这两个构型上，可得

$$\begin{aligned}
&\int_{\substack{\overline{\xi}(x)=\overline{\eta}(x),\\ \forall x\in\Lambda}} \mathcal{D}\overline{\xi} \int_{\substack{\xi(x)=\eta(x),\\ \forall x\in\Lambda}} \mathcal{D}\xi D\left(\xi,\overline{\xi};\lambda\right) \\
&= \int \mathcal{D}\overline{\xi}\mathcal{D}\xi D\left(\xi,\overline{\xi};\lambda\right) \prod_{x\in\Lambda} \delta\left(\overline{\xi}(x)-\overline{\eta}(x)\right)\delta\left(\xi(x)-\eta(x)\right) \\
&= \int \mathcal{D}\overline{\xi}\mathcal{D}\xi D\left(\xi,\overline{\xi};\lambda\right) \prod_{t\in\mathcal{T}}\prod_{\vec{x}\in\mathcal{X}} \delta\left(\overline{\xi}(t,\vec{x})-\overline{\eta}(t,\vec{x})\right)\delta\left(\xi(t,\vec{x})-\eta(t,\vec{x})\right) \\
&= \left\langle\Psi\left|V(0,\Delta t)^{\dagger} O^{\dagger}_{\overline{\eta},\Lambda} O_{\eta,\Lambda} V(0,\Delta t)\right|\Psi\right\rangle, \qquad (20)
\end{aligned}$$

where, for convenience, we have defined

其中为方便起见，我们定义了

$$O_{\eta,\Lambda} := T\left\{\prod_{t\in\mathcal{T}} V(t,t+\Delta t) \prod_{\vec{x}\in\mathcal{X}} \delta\left(\phi(t,\vec{x})-\eta(t,\vec{x})\right)\right\}. \qquad (21)$$

Now, using the form of the interaction unitary in (15), we see that for small enough Δt , we have

现在利用 (15) 中相互作用幺正算符的形式可以看出，当 Δt 足够小时，我们有

$$V(t,t+\Delta t) \approx \exp\left(-i\Delta t H^{(I)}_{\text{int}}(t)\right). \qquad (22)$$

Assuming a ϕ^4 self-interaction, $H^{(I)}_{\text{int}}(t)$ is given in (15). For small spatial lattice spacing, $\Delta x, H^{(I)}_{\text{int}}(t)$ is approximately

对于 ϕ^4 自相互作用，$H^{(I)}_{\text{int}}(t)$ 由 (15) 给出。当空间格距很小时，$\Delta x, H^{(I)}_{\text{int}}(t)$ 近似为

$$H_{\text{int}}^{(I)}(t) \approx -\sum_{\vec{x}\in\mathcal{X}} (\Delta x)^{d-1} \frac{\lambda}{4!} \phi\big(t,\vec{x}\big)^4, \tag{23}$$

which allows us to write

由此我们可以写出

$$\begin{aligned} V(t,t+\Delta t) &\approx \exp\left(i\Delta t(\Delta x)^{d-1} \frac{\lambda}{4!} \sum_{\vec{x}\in\mathcal{X}} \phi\big(t,\vec{x}\big)^4 \right) \\ &= \prod_{\vec{x}\in\mathcal{X}} e^{i\Delta V \frac{\lambda}{4!} \phi(t,\vec{x})^4} \end{aligned} \tag{24}$$

where we have defined the lattice volume element $\Delta V = \Delta t(\Delta x)^{d-1}$. The last line follows from the fact that, for distinct $\vec{x}$ and $\vec{x}'$, we have $\big[\phi\big(t,\vec{x}\big),\phi\big(t,\vec{x}'\big)\big] = 0$, and hence $e^{a\phi(t,\vec{x})+b\phi(t,\vec{x}')} = e^{a\phi(t,\vec{x})}e^{b\phi(t,\vec{x}')}$ for constants a and b .

其中我们定义了格点体积元 $\Delta V = \Delta t(\Delta x)^{d-1}$ 。最后一行由以下事实推导而来: 对于不同的 $\vec{x}$ 和 $\vec{x}'$ ，我们有 $\big[\phi\big(t,\vec{x}\big),\phi\big(t,\vec{x}'\big)\big] = 0$ ，因此对于常数 a 和 b 可得 $e^{a\phi(t,\vec{x})+b\phi(t,\vec{x}')} = e^{a\phi(t,\vec{x})}e^{b\phi(t,\vec{x}')}$ 。

Further, we can reorder any of the delta functions in the product $\prod_{\vec{x}\in\mathcal{X}} \delta(\phi\big(t,\vec{x}\big) - \eta\big(t,\vec{x}\big))$ in (21), since they all commute. Replacing $V(t,t+\Delta t)$ in (21) by the last line of (24), we can also distribute the terms $e^{i\Delta V \frac{\lambda}{4!}\phi(t,\vec{x})^4}$ from $V(t,t+\Delta t)$ throughout the product of delta functions in any order we want, since everything commutes. This means we can rewrite

此外，由于所有 δ 函数都对易，我们可以对乘积 (21) 式中 $\prod_{\vec{x}\in\mathcal{X}} \delta(\phi\big(t,\vec{x}\big) - \eta\big(t,\vec{x}\big))$ 的任意 δ 函数重新排序。将 (21) 式中的 $V(t,t+\Delta t)$ 替换为 (24) 式的最后一行，又由于所有元素都对易，我们可以按任意想要的顺序，将来自 $V(t,t+\Delta t)$ 的项 $e^{i\Delta V \frac{\lambda}{4!}\phi(t,\vec{x})^4}$ 分配到 δ 函数的整个乘积中。这意味着我们可以将其重写为

$$\begin{aligned} O_{\eta,\Lambda} &\approx T\left\{ \prod_{t\in\mathcal{T}} \prod_{\vec{x}\in\mathcal{X}} e^{i\Delta V \frac{\lambda}{4!}\phi(t,\vec{x})^4} \delta\left(\phi\left(t,\vec{x}\right) - \eta\left(t,\vec{x}\right)\right) \right\} \\ &= C\left\{ \prod_{x\in\Lambda} e^{i\Delta V \frac{\lambda}{4!}\phi(x)^4} \delta\left(\phi(x) - \eta(x)\right) \right\}, \end{aligned} \tag{25}$$

where in the last line we replaced the double product with a single product over lattice points. For the purpose of comparison with certain causal set expressions in sections ”Histories Framework” and ”Introducing an Interaction”, we have also replaced the time ordering with a causal ordering, denoted by $C\{\cdot\}$. For a product of r operators $O_1(x_1)\ldots O_r(x_r)$, at the spacetime points $x_1,\ldots,x_r$ respectively, $C\{O_1(x_1)\ldots O_r(x_r)\}$ means we order the operators in any way that is consistent with the following rule: if $x_n \preccurlyeq x_m$ (x_n is to the causal past of x_m), then $O_n(x_n)$ cannot appear to the left of $O_m(x_m)$ (Note that any choice of order satisfying this rule can be related to any other satisfactory choice by the permutation of operators localized at spacelike points. By the commutativity of spacelike operators, this therefore does not change the total operator in question, and hence any satisfactory choice is equivalent.). We are free to replace the time ordering by a causal ordering in (25) as any causal ordering can be modified into a time ordering by permuting operators at spacelike points, which commute by spacelike commutativity.

其中我们在最后一行将二重乘积替换为格点上的单乘积。为了和“历史框架”与“引入相互作用”两节中的某些因果集表达式作对比，我们还将时间序替换为因果序，记为 $C\{\cdot\}$。对于分别位于时空点 $x_1, \ldots, x_r$ 处的 r 个算符 $O_1(x_1) \ldots O_r(x_r)$，$C\{O_1(x_1) \ldots O_r(x_r)\}$ 表示我们按照符合以下规则的方式对算符排序: 如果 $x_n \preccurlyeq x_m$ (即 x_n 在 x_m 的因果过去)，那么 $O_n(x_n)$ 不能出现在 $O_m(x_m)$ 的左侧 (注意，任何满足该规则的排序选择都可以通过交换类空点处局域化算符得到其他满足规则的排序。由类空算符的对易性，这不会改变我们讨论的总算符，因此任何满足要求的选择都是等价的)。我们可以自由地将 (25) 式中的时间序替换为因果序，这是因为任何因果序都可以通过交换类空点处的算符转化为时间序，而类空间算符满足对易性。

If we allow ourselves some more mathematical slack for the time being, we can imagine taking (20) "further" by pinning the configurations ξ and $\bar{\xi}$ to η and $\bar{\eta}$ for every spacetime point $x \in M$, rather than just for those in our lattice Λ. The double path integral in the first line of (20) is then no longer an integral and simply evaluates to $D(\eta, \bar{\eta}; \lambda)$ - the decoherence functional evaluated at η and $\bar{\eta}$. For convenience, let us relabel $\eta, \bar{\eta} \mapsto \xi, \bar{\xi}$. Now, following (20), and the last line of (25), we are motivated to write down the following formal expression for $D\left(\xi, \bar{\xi}; \lambda\right)$:

如果我们暂时容许在数学上稍作放宽，我们可以想象对 (20) 进行“进一步”拓展: 将场构型 ξ 和 $\bar{\xi}$ 固定在每个时空点 $x \in M$ 处的 η 和 $\bar{\eta}$，而非仅固定在格点 Λ 处。这样一来，(20) 第一行中的双路径积分就不再是积分，直接计算得到 $D(\eta, \bar{\eta}; \lambda)$ ——即取值在 η 和 $\bar{\eta}$ 处的退相干泛函。为方便起见，我们对 $\eta, \bar{\eta} \mapsto \xi, \bar{\xi}$ 重新标记。现在，遵循 (20) 和 (25) 的最后一行，我们受启发为 $D\left(\xi, \bar{\xi}; \lambda\right)$ 写下如下形式表达式:

$$
\begin{aligned}
D\left(\xi, \bar{\xi}; \lambda\right) = \langle \Psi \mid & C\left\{ \prod_{x \in M} e^{idV \frac{\lambda}{4!} \phi(x)^4} \delta\left(\phi(x) - \bar{\xi}(x)\right) \right\}^{\dagger} \\
& C\left\{ \prod_{x \in M} e^{idV \frac{\lambda}{4!} \phi(x)^4} \delta\left(\phi(x) - \xi(x)\right) \right\} |\Psi\rangle,
\end{aligned} \tag{26}
$$

where dV is the infinitesimal volume element. The free theory is given by $\lambda = 0$, and thus for the free decoherence functional, we have the following formal expression:

其中 dV 是无穷小体积元。自由场论由 $\lambda = 0$ 给出，因此对于自由退相干泛函，我们得到如下形式表达式:

$$
D\left(\xi, \bar{\xi}\right) = \left\langle \Psi \middle| C\left\{ \prod_{x \in M} \delta\left(\phi(x) - \bar{\xi}(x)\right) \right\}^{\dagger} C\left\{ \prod_{x \in M} \delta\left(\phi(x) - \xi(x)\right) \right\} \middle| \Psi \right\rangle. \tag{27}
$$

We call these expressions "formal" as the products over all spacetime points $x \in M$ are not well defined in the continuum, but products over all causal set points can be defined, as we will see below. For the sake of comparison with certain causal set expressions in sections "Histories Framework" and "Introducing an Interaction" (specifically (35) and (43)), it is important to again highlight that the field operators, $\phi(x)$, in these expressions are in the interaction (Heisenberg) picture in the interacting (free) theory.

我们称这些表达式是“形式上的”，因为对所有时空点 $x \in M$ 的乘积在连续统中没有良好定义，但对所有因果集点的乘积是可以定义的，我们下文会说明。为了和“历史框架”小节与“引入相互作用”小节中的特定因果集表达式 (具体是 (35) 和 (43)) 作比较，需要再次强调: 这些表达式中的场算符 $\phi(x)$ 在相互作用 (自由) 理论中处于相互作用 (海森堡) 绘景。

Recalling the definition of $D\left(\xi, \bar{\xi}; \lambda\right)$ in Equation (13), Equation (26) is then telling us how to write the rhs of (13) in terms of the usual quantities of the canonical framework, e.g., field operators and the given state $|\Psi\rangle$. Equation (27) is similarly doing this for the free theory. For the causal set case in sections ”Histories Framework” and ”Introducing an Interaction”, we will write down an analogous equality (for both the free and interacting theories) between an explicit measure on ξ and $\bar{\xi}$ and expressions in the canonical framework.

回顾 (13) 式中 $D\left(\xi, \bar{\xi}; \lambda\right)$ 的定义，(26) 式告诉我们如何用正则框架中的常规量 (例如场算符和给定态 $|\Psi\rangle$) 写出 (13) 式的右侧。(27) 式同样为自由场论完成了这一步。对于“历史框架”小节和“引入相互作用”小节中的因果集情形，我们会在 ξ 和 $\bar{\xi}$ 上的显式测度与正则框架中的表达式之间，写下 (对自由理论和相互作用理论都成立的) 一个类似等式。

Causal Set Free Scalar Field Theory

因果集自由标量场论

Canonical Framework

正则框架

Here, we follow the Sorkin-Johnston formalism for real scalar QFT [2, 6, 7]. The signature of this formalism is its choice of ground state of the quantum theory. This choice is made using the positive part of the Pauli-Jordan function. We will see precisely what this means shortly. First, let us introduce some basic concepts for causal set QFT.

本文遵循针对实标量量子场论的 Sorkin-Johnston 形式体系 [2, 6, 7]。该形式体系的核心特征是对量子理论基态的选取，该选取基于泡利-约当函数的正部实现。我们很快就会明确这一点，首先先介绍因果集量子场论的一些基础概念。

Consider a fixed causal set $(C, \preccurlyeq)$ of finite cardinality $N = |C|$ with some natural labelling. That is, if $x, y \in C$ are two causal set points with the labels $i, j \in \{1, \dots, N\}$, respectively, then if $x \preccurlyeq y$, then $i \leq j$ (we only have equality if $x = y$). In what follows, we will simplify our notation by using x to denote both the causal set point and its label. Thus, we will write statements like $x \preccurlyeq y$ and $x \leq y$. It should be clear from the context whether the point or the label is being referred to. We will also use the point and its label interchangeably in subscripts/superscripts of matrices and vectors.

考虑一个具有基数 $N = |C|$ 、带自然标记的固定因果集 $(C, \preccurlyeq)$ 。也就是说，若 $x, y \in C$ 是两个分别带有标记 $i, j \in \{1, \ldots, N\}$ 的因果集点，那么如果 $x \preccurlyeq y$ ，则有 $i \leq j$ (仅当 $x = y$ 时取等)。在下文中我们简化记号，使用 x 同时指代因果集点及其标记，因此我们可以写出 $x \preccurlyeq y$ 和 $x \leq y$ 这类表述，根据上下文可以清晰区分我们指代的是点还是标记，我们也会在矩阵和向量的下标/上标中互换使用点和它的标记。

We introduce a discreteness length scale, l_0 , which is assumed to be of order the Planck length. Through this, we define a discreteness density $\rho = 1/l_0^d$, where the dimension d must also be chosen for comparison with the continuum theory of the same dimension. In the case that C comes from a sprinkling into a d -dimensional spacetime of finite volume, ρ is the sprinkling density.

我们引入离散长度标度 l_0 ，一般认为其量级为普朗克长度。由此我们定义离散密度 $\rho = 1/l_0^d$ ，为了和同维度的连续体理论对比，还需要选定维度 d 。当 C 由喷淋过程生成到有限体积的 d 维时空中时，ρ 就是喷淋密度。

We begin with a retarded Green function, G_{xy} , where by "retarded" we mean that $G_{xy} = 0$ unless $y \preccurlyeq x$. This is analogous to the continuum retarded Green function $G(x, y)$, which is a Green function of the Klein-Gordon equation $(\Box + m^2)\varphi = 0$. There are different candidates for the retarded Green function on a causal set. For example, one can lift G_{xy} from the continuum retarded Green function $G(x, y)$ using the embedding of the causal set into the continuum spacetime [8]. Alternatively, one can define G_{xy} through a "hop-and-stop" model as in [9]. Here, we will consider the latter case for concreteness.

我们从推迟格林函数 G_{xy} 开始研究，所谓“推迟”指的是除非 $y \preccurlyeq x$ 成立，否则 $G_{xy} = 0$ 为零。这对应连续体中的推迟格林函数 $G(x, y)$ ，它是克莱因-戈登方程 $(\Box + m^2)\varphi = 0$ 的格林函数。因果集上的推迟格林函数有多种候选形式: 例如可以通过将因果集嵌入连续体时空，从连续体推迟格林函数 $G(x, y)$ 提升得到 G_{xy} [8]，也可以像文献 [9] 中那样通过“跳停”模型定义 G_{xy} 。为明确起见，本文讨论后一种情况。

Note that for a globally hyperbolic continuum spacetime, the advanced Green function equals the transpose of the retarded Green function [7], and so we take the advanced Green function on the causal set to be $G^T_{xy} = G_{yx}$. We then define the Pauli-Jordan matrix $\Delta := G - G^T$, where we have omitted the indices for brevity. It is straightforward to see, then, that the matrix $\frac{1}{\rho} i\Delta$ is Hermitian (the inclusion of $1/\rho$ here allows us to directly compare the eigenvalues with those of the continuum Pauli-Jordan function [10]), and thus we can find a complete orthonormal basis of (complex-valued) eigenvectors and their associated (real-valued) eigenvalues. Here, we have assumed the standard inner product on $\mathbb{C}^N$, i.e., $(v, w) = \sum_{x \in C} v^*_x w_x$.

注意，对于整体双曲的连续体时空，推迟格林函数的转置就是超前格林函数 [7]，因此我们取因果集上的超前格林函数为 $G^T_{xy} = G_{yx}$ 。接下来我们定义泡利-约当矩阵 $\Delta := G - G^T$ ，为简洁起见省略了指标。不难看出矩阵 $\frac{1}{\rho} i\Delta$ 是埃尔米特矩阵 (此处引入 $1/\rho$ 可以让我们直接将本征值和连续体泡利-约当函数的本征值对比 [10])，因此我们可以找到一组由 (复值) 本征向量构成的完全标准正交基，以及对应的 (实值) 本征值。此处我们默认使用 $\mathbb{C}^N$ 上的标准内积，即 $(v, w) = \sum_{x \in C} v^*_x w_x$ 。

We further find that if a vector v has a non-zero eigenvalue μ , then the complex conjugate of the vector, v^* , has eigenvalue $-\mu$. Thus, we can group eigenvectors based on whether their eigenvalues are positive, negative, or zero. Let $K < N$ be the number of positive eigenvalues, and let $v_x^{(k)}$ be the components of a normalized eigenvector $v^{(k)}$ with positive eigenvalue $\mu_k > 0$ (where $k = 1, \ldots, K$ labels the positive eigenvalues). That is,

我们进一步发现，若向量 v 存在非零本征值 μ ，则该向量的复共轭向量 v^* 对应的本征值为 $-\mu$ 。因此，我们可以根据本征值为正、负或零对本征向量分组。设 $K < N$ 为正本征值的数量，$v_x^{(k)}$ 是带有正本征值 $\mu_k > 0$ 的归一化本征向量 $v^{(k)}$ 的分量 (其中 $k = 1, \ldots, K$ 标记正本征值)。即

$$\frac{1}{\rho} \sum_{y \in C} i\Delta_{xy} v_y^{(k)} = \mu_k v_x^{(k)}. \tag{28}$$

The components of the K negative eigenvalue eigenvectors are then $\left(v_x^{(k)}\right)^*$. The remaining $N - 2K$ eigenvectors have eigenvalue 0 and form a basis of the kernel $\ker \Delta$. We can now write

K 个负本征值本征向量的分量为 $\left(v_x^{(k)}\right)^*$ ，剩余 $N - 2K$ 个本征向量的本征值为 0，构成核空间 $\ker \Delta$ 的一组基。现在我们可以写出

$$i\Delta_{xy} = \rho \sum_{k=1}^{K} \mu_k \left(v_x^{(k)} \left(v_y^{(k)}\right)^* - \left(v_x^{(k)}\right)^* v_y^{(k)} \right). \tag{29}$$

Note, by thinking of ρ^{-1} as the discrete volume element ΔV , we see that (28) is analogous to the continuum integral eigenequation

注意，将 ρ^{-1} 视为离散体积元 ΔV 后，我们可以发现式 (28) 与连续统积分本征方程类似

$$\int_M dV_y i\Delta(x, y)\, v^{(k)}(y) = \mu_k v^{(k)}(x), \tag{30}$$

for some finite volume spacetime M . We can then contrast the causal set eigenvalues directly to their continuum counterparts. Here, dV_y is the usual spacetime volume element, and $\Delta(x, y)$ is the continuum Pauli-Jordan function, defined in terms of the retarded Green function $G(x, y)$ as $\Delta(x, y) = G(x, y) - G(y, x)$.

该方程适用于某个有限体积时空 M 。我们可以直接将因果集的本征值与连续统对应本征值进行对比。此处，dV_y 是常规时空体积元，$\Delta(x, y)$ 是连续统泡利-约当函数，它根据推迟格林函数 $G(x, y)$ 定义为 $\Delta(x, y) = G(x, y) - G(y, x)$ 。

Returning to causal set QFT, for each positive eigenvalue eigenvector of $i\Delta$, we introduce annihilation operators a_k (where $k = 1, \ldots, K$) satisfying the commutation relations $\left[a_k, a_l^\dagger\right] = \delta_{kl}\mathbb{1}$. The signature step of the Sorkin-Johnston formalism is then to define the normalized ground state, $|\Omega\rangle$, also known as the Sorkin-Johnston state, as that which satisfies $a_k |\Omega\rangle = 0$ for all $k = 1, \ldots, K$. We | further define orthonormal basis states as $|n_1, \ldots, n_K\rangle = \prod_{k=1}^{K} \frac{\left(a_k^\dagger\right)^{n_k}}{\sqrt{n_k!}} |\Omega\rangle$, where $n_k = 0, 1, 2, \ldots$ The Fock space of our theory, $\mathcal{F}$, is then the span of these states and is isomorphic to the (completion of) the tensor product of K harmonic oscillator Hilbert spaces.

回到因果集量子场论，对于 $i\Delta$ 的每个正本征值本征向量，我们引入湮灭算符 a_k (其中 $k = 1, \ldots, K$)，满足对易关系 $\left[a_k, a_l^\dagger\right] = \delta_{kl}\mathbb{1}$ 。索金-约翰斯顿形式体系的关键步骤是定义归一化基态 $|\Omega\rangle$，也即索金-约翰斯顿态，它满足 $a_k|\Omega\rangle = 0$ for all $k = 1, \ldots, K$. We | ；我们进一步将标准正交基态定义为 $|n_1, \ldots, n_K\rangle = \prod_{k=1}^{K} \frac{\left(a_k^\dagger\right)^{n_k}}{\sqrt{n_k!}}|\Omega\rangle$ ，其中 $n_k = 0, 1, 2, \ldots$ 。我们理论的福克空间 $\mathcal{F}$ 由这些态张成，同构于 K 个谐振子希尔伯特空间张量积的 (完备化)。

For each point $x \in C$, we define the field operator at x as

对于每个点 $x \in C$ ，我们将 x 处的场算符定义为

$$\phi_x = \sqrt{\rho}\sum_{k=1}^{K}\sqrt{\mu_k}\left(v_x^{(k)}a_k + \left(v_x^{(k)}\right)^* a_k^\dagger\right). \tag{31}$$

This expansion in terms of modes is comparable to the continuum expansion of the field operator-valued distribution $\phi(x)$ in terms of plane waves, e.g., $\phi(x) = \sum_k u^{(k)}(x)\, a_k + u^{(k)}(x)^* a_k^\dagger$. In the free/interacting continuum theory, expanding $\phi(x)$ in this way, i.e., in terms of modes $u^{(k)}(x)$ that satisfy the free equations of motion, tells us that $\phi(x)$ is in the Heisenberg/Interaction picture, as it carries the free dynamics. Thus, we should think of our causal set field operator, ϕ_x , as being in the Heisenberg/Interaction picture in the free/interacting causal set QFT.

这种按模展开类似于场算符值分布 $\phi(x)$ 按平面波的连续展开，例如 $\phi(x) = \sum_k u^{(k)}(x)\, a_k + u^{(k)}(x)^* a_k^\dagger$ 。在自由/相互作用连续统理论中，以这种方式，即按满足自由运动方程的模 $u^{(k)}(x)$ 展开 $\phi(x)$，说明 $\phi(x)$ 处于海森堡绘景/相互作用绘景，因为它承载自由动力学。因此，我们应将我们的因果集场算符 ϕ_x 视为自由/相互作用因果集量子场论中处于海森堡绘景/相互作用绘景的算符。

Using (31), one can then verify that

利用 (31)，我们可以验证

$$\left[\phi_x, \phi_y\right] = i\Delta_{xy}\mathbb{1}, \tag{32}$$

which is analogous to the continuum covariant commutation relations $[\phi(x), \phi(y)] = i\Delta(x, y)$. As in the continuum, if $x, y \in C$ are mutually spacelike, then $\Delta_{xy} = 0$, and hence ϕ_x commutes with ϕ_y . In this precise sense, we can think of the field operator, ϕ_x , as ”local” to x .

这与连续统协变对易关系 $[\phi(x), \phi(y)] = i\Delta(x, y)$ 类似。和连续统情况一样，如果 $x, y \in C$ 互为类空分离，则 $\Delta_{xy} = 0$ ，因此 ϕ_x 与 ϕ_y 对易。在这个精确意义上，我们可以将场算符 ϕ_x 视为对 x “定域”的算符。

In addition, from (31), we can also determine the 2-point, or Wightman, function of our theory in terms of our modes:

此外，由 (31)，我们还可以用我们的模确定理论的两点函数，即怀特曼函数:

$$W_{xy} = \langle \Omega | \phi_x \phi_y | \Omega \rangle = \rho \sum_{k=1}^{K} \mu_k v_x^{(k)} \left(v_y^{(k)} \right)^* . \tag{33}$$

Recalling (29) we see that this 2-point function is simply the positive part of $i\Delta$.

回顾 (29) 我们可知，这个两点函数正是 $i\Delta$ 的正分量。

One aspect not yet accounted for is in what sense the causal set field operators satisfy the ”equations of motion,” which is a standard requirement of the continuum operator-valued distribution $\phi(x)$. In fact, the expansion (31) takes this into account in a way that is analogous to the continuum case.

一个尚未说明的问题是, 因果集场算符在何种意义上满足 “运动方程”, 这是连续统算符值分布 $\phi(x)$ 的标准要求。事实上，展开式 (31) 已经以类似于连续统的方式考虑了这一点。

In the continuum, a nice family of classical solutions are those of compact support on Cauchy surfaces. For any such solution, $\left(\Box + m^2\right)\varphi(x) = 0$, we can find a test function (a smooth function whose support in spacetime is compact), $f(x)$, such that

在连续统中, 一类良好的经典解是柯西面上具有紧支集的解。对任意这类解 $\left(\Box + m^2\right)\varphi(x) = 0$, 我们可以找到一个测试函数 (时空支集为紧集的光滑函数) $f(x)$ ， 使得

$$\varphi(x) = \int_M dV_y \Delta(x, y) f(y) . \tag{34}$$

We also write this more compactly as $\varphi = \Delta f$. One can further show that $\ker(\Box + m^2) = \operatorname{im}\Delta$ [7, 11, 12] . Given the L^2 -inner product on complex-valued functions on the spacetime M (where we take M to be of finite volume for simplicity), $(\varphi, \xi)_{L^2} = \int_M dV_x \varphi(x)^* \eta(x)$, we then find that solutions are orthogonal to functions in the kernel of Δ .

我们也可以将其更简洁地写为 $\varphi = \Delta f$ 。进一步可以证明 $\ker(\Box + m^2) = \operatorname{im}\Delta$ [7, 11, 12] 。时空 M (为简便我们取 M 为有限体积) 上的复值函数带有 L^2 内积 $(\varphi, \xi)_{L^2} = \int_M dV_x \varphi(x)^* \eta(x)$ ，据此我们发现，解与 Δ 核中的函数正交。

In the causal set case, we can analogously define a ”solution” as a complex-valued function on C that is orthogonal to $\ker\Delta$ or equivalently $\ker i\Delta$. Given the expansion of ϕ_x (31) in terms of non-zero eigenvalue eigenvectors of $i\Delta$, it is straightforward to verify that, for any $w \in \ker i\Delta$, we have $\sum_{x\in C} w_x^* \phi_x = 0$, i.e., ϕ_x is orthogonal to $\ker i\Delta$ as desired.

在因果集情形, 我们可以类似地将 “解” 定义为 C 上与 $\ker\Delta$ 正交, 等价于与 $\ker i\Delta$ 正交的复值函数。已知 ϕ_x 按 $i\Delta$ 非零本征值本征向量的展开式 (31)， 不难验证对任意 $w \in \ker i\Delta$ 都有 $\sum_{x\in C} w_x^* \phi_x = 0$ ，即 ϕ_x 按要求与 $\ker i\Delta$ 正交。

Remark. It is worth commenting on the choice of modes and ladder operators implicit in the expansion (31) and thus the choice of the (Gaussian) Sorkin-Johnston ground state $|\Omega\rangle$. This choice also led to the specific 2-point function in (33). Alternatively, we could have followed algebraic QFT and introduced abstract algebra elements, ϕ_x , for each $x \in C$, satisfying (32) and orthogonal to keri Δ as above. After this, we could then pick any sufficiently well-behaved 2-point function W' (By sufficiently well behaved, we mean that W' is Hermitian and, given any $v \in \ker\Delta, W.v = 0$ and $v^{\dagger}.W = 0$.), and use the Gel' fand-Naimark-Segal (GNS) representation theorem to represent the algebra of field operators on some Hilbert space, and in the process define the Gaussian ground state in this Hilbert space as that which has the 2-point function W' . This ground state need not coincide with the SJ state $|\Omega\rangle$ above.

注: 对展开式 (31) 中隐含的模式与阶梯算符的选择，即 (高斯型) 索尔金-约翰斯顿基态 $|\Omega\rangle$ 的选择，是值得讨论的。这一选择也导出了式 (33) 中特定的两点关联函数。我们也可以转而沿用代数量子场论的思路，对每个 $x \in C$ 引入抽象代数元 ϕ_x ，令其满足条件 (32) 且与上文的 keri Δ 正交。在此之后，我们可以任选一个性质足够良好的两点关联函数 W' (性质足够良好指 W' 是厄米的，且对任意 $v \in \ker\Delta, W.v = 0$ 与 $v^{\dagger}.W = 0$ 都满足要求)，再利用盖尔范德-奈马克-西格尔 (GNS) 表示定理将场算子代数表示在某个希尔伯特空间上，在此过程中将该希尔伯特空间中具有两点关联函数 W' 的态定义为高斯基态。该基态不必与上述 SJ 态 $|\Omega\rangle$ 一致。

Histories Framework

历史框架

With the canonical framework in place, we are now ready to define the decoherence functional for a free real scalar field on a fixed causal set, $(C, \preccurlyeq)$, of finite cardinality $N = |C|$. Recall that we have assumed a natural labelling of C . That is, for any pair of distinct points $x, y \in C$, if $x \preccurlyeq y$ as points, then $x < y$ as labels.

有了正则框架之后，我们现在可以着手定义有限基数 $N = |C|$ 的固定因果集 $(C, \preccurlyeq)$ 上自由实标量场的退相干泛函了。回顾一下，我们已经对 C 做了自然标号。也就是说，对任意一对不同的点 $x, y \in C$ ，若 $x \preccurlyeq y$ 满足点之间的因果关系，则 $x < y$ 满足标号之间的序关系。

Now, given the formal continuum expression in (27), we are motivated to define the free decoherence functional as Sorkin did in [3]:

现在，基于式 (27) 的形式连续统表达式，我们仿照 Sorkin 在文献 [3] 中的工作定义自由退相干泛函如下:

$$D\left(\xi, \bar{\xi}\right) = \left\langle \Omega \middle| \delta\left(\phi_1 - \bar{\xi}_1\right) \ldots \delta\left(\phi_N - \bar{\xi}_N\right) \delta\left(\phi_N - \xi_N\right) \ldots \delta\left(\phi_1 - \xi_1\right) \middle| \Omega \right\rangle, \tag{35}$$

where ξ and $\bar{\xi}$ are real-valued field configurations on C , with values ξ_x and $\bar{\xi}_x$ at a given point $x \in C$.

其中 ξ 和 $\bar{\xi}$ 是 C 上的实值场构型，在给定点 $x \in C$ 处的场值分别为 ξ_x 和 $\bar{\xi}_x$ 。

The ordering of the delta functions according to their natural labelling means we have already implemented the causal ordering of (27). It is worth pointing out that, unlike the uncountably many delta functions in rhs of (27), the finite number of delta functions here poses no problem to the well-definedness of this decoherence functional as a measure over $\mathbb{R}^{2N}$, i.e., over the space of all field configurations ξ and $\bar{\xi}$.

按照自然标号对 delta 函数排序，意味着我们已经实现了式 (27) 要求的因果序。值得指出的是，和式 (27) 右侧不可数无穷多个 delta 函数不同，这里有限个 delta 函数不会导致退相干泛函作为 $\mathbb{R}^{2N}$ (即所有场构型 ξ 和 $\bar{\xi}$ 构成的空间) 上的测度出现良定义性问题。

One of the main results of [3] is that the rhs of (35) can be manipulated into a form analogous to the continuum expression in (12). One finds

文献 [3] 的一个主要结论是，式 (35) 的右侧可以整理为和式 (12) 中连续统表达式类似的形式，可得：

$$D\left(\xi,\bar{\xi}\right)=\mathcal{N}e^{i\Delta S\left[\xi,\bar{\xi}\right]}\delta\left(G^T\left(\xi-\bar{\xi}\right)\right)e^{-Q\left[\xi,\bar{\xi}\right]}, \tag{36}$$

where $\Delta S\left[\xi,\bar{\xi}\right]$ is a bilinear form in ξ and $\bar{\xi}$ and is analogous to $S\left[\xi\right]-S\left[\bar{\xi}\right]$ in the continuum expression (12). $Q\left[\xi,\bar{\xi}\right]$ is also a bilinear form and depends upon the symmetric part of the SJ 2-point function W . In this sense, the term $e^{-Q\left[\xi,\bar{\xi}\right]}$ takes into account the choice of ”initial” state; cf. the term $\Psi\left(\bar{\xi}\left(0\right)\right)^*\Psi\left(\xi\left(0\right)\right)$ in (12). $\delta\left(G^T\left(\xi-\bar{\xi}\right)\right)$ is a delta function ensuring that $\xi-\bar{\xi}$ is in the kernel of G^T (the advanced Green function). This is analogous to the delta function $\delta\left(\xi\left(T\right)-\bar{\xi}\left(T\right)\right)$ in (12), which ensures the two field configurations are ”pinned together” on the future boundary. The delta function $\delta\left(G^T\left(\xi-\bar{\xi}\right)\right)$, in some sense, does the same job (see [3] for further discussion). Finally, there is a normalization constant, $\mathcal{N}$, at the front to ensure that $\int_{\mathbb{R}^{2N}}d^N\xi d^N\bar{\xi}D\left(\xi,\bar{\xi}\right)=1$. For the explicit forms of ΔS and Q , and for further comparisons to their continuum counterparts, see [3].

其中 $\Delta S\left[\xi,\bar{\xi}\right]$ 是 ξ 和 $\bar{\xi}$ 构成的双线性型，对应连续统表达式 (12) 中的 $S\left[\xi\right]-S\left[\bar{\xi}\right]$。$Q\left[\xi,\bar{\xi}\right]$ 同样是一个双线性型，依赖于 SJ 两点函数 W 的对称部分。从这个意义上说，项 $e^{-Q\left[\xi,\bar{\xi}\right]}$ 已经考虑了初态的选择，对应式 (12) 中的项 $\Psi\left(\bar{\xi}\left(0\right)\right)^*\Psi\left(\xi\left(0\right)\right)$。$\delta\left(G^T\left(\xi-\bar{\xi}\right)\right)$ 是一个 delta 函数，它保证 $\xi-\bar{\xi}$ 属于超前格林函数 G^T 的核，这对应式 (12) 中保证两个场构型在未来边界“粘合”的 delta 函数 $\delta\left(\xi\left(T\right)-\bar{\xi}\left(T\right)\right)$。从某种意义上说，delta 函数 $\delta\left(G^T\left(\xi-\bar{\xi}\right)\right)$ 起到了相同的作用 (进一步讨论见文献 [3])。最后，式前有归一化常数 $\mathcal{N}$ ，用于保证 $\int_{\mathbb{R}^{2N}}d^N\xi d^N\bar{\xi}D\left(\xi,\bar{\xi}\right)=1$。关于 ΔS 和 Q 的具体形式，以及它们和连续统对应形式的进一步对比，参见文献 [3]。

One aspect of (36) mentioned in [3], and still unexplored, is how one might recover the continuum expression in (12) via some continuum limit ($N\to\infty$) of (36).

文献 [3] 提到了式 (36) 的一个方面，目前仍未得到探索，即如何通过式 (36) 的连续统极限 ($N\to\infty$) 得到式 (12) 的连续统表达式。

Causal Set Interacting Scalar Field Theory

因果集相互作用标量场理论

Introducing an Interaction

引入相互作用

We now introduce a ϕ^4 self-interaction to our real scalar field theory. In the continuum this is done by adding an interaction term to the action, $S_{\text{int}}[\xi;\lambda] = \int_M dV \frac{\lambda}{4!}\xi^4$, which, following (13), results in the interacting decoherence functional $D\left(\xi,\bar{\xi};\lambda\right) = D\left(\xi,\bar{\xi}\right)e^{i\frac{\lambda}{4!}\int_M dV\left(\xi^4-\bar{\xi}^4\right)}$. In the causal set case, we are then motivated to modify the term $\Delta S\left[\xi,\bar{\xi}\right]$ in the free decoherence functional to be

我们现在为实标量场理论引入一个 ϕ^4 自相互作用。在连续统中，这是通过给作用量添加一个相互作用项 $S_{\text{int}}[\xi;\lambda] = \int_M dV \frac{\lambda}{4!}\xi^4$ 实现的，根据式 (13)，由此可得退相干泛函为 $D\left(\xi,\bar{\xi};\lambda\right) = D\left(\xi,\bar{\xi}\right)e^{i\frac{\lambda}{4!}\int_M dV\left(\xi^4-\bar{\xi}^4\right)}$。在因果集情形中，我们受此启发将自由退相干泛函中的项 $\Delta S\left[\xi,\bar{\xi}\right]$ 修改为

$$\Delta S\left[\xi,\bar{\xi};\lambda\right] = \Delta S\left[\xi,\bar{\xi}\right] + \frac{1}{\rho}\sum_{n=1}^{N}\frac{\lambda_n}{4!}\left(\xi_n^4 - \bar{\xi}_n^4\right), \tag{37}$$

where we recall that $\frac{1}{\rho}$ is like a discrete volume element ΔV. Here, $\lambda \in \mathbb{R}^N$ and has components λ_n. Allowing λ to vary across the causal set C gives us the freedom to describe interactions local to some region of spacetime.

其中我们回顾，$\frac{1}{\rho}$ 类似于离散体积元 ΔV。此处，$\lambda \in \mathbb{R}^N$，其分量为 λ_n。允许 λ 在整个因果集 C 上变化，为我们提供了描述时空特定区域局域相互作用的自由度。

With this, we now have a candidate decoherence functional for an interacting real scalar field theory on a causal set:

至此，我们就得到了因果集上相互作用实标量场理论的一个候选退相干泛函:

$$\begin{aligned} D\left(\xi,\bar{\xi};\lambda\right) &= \mathcal{N}(\lambda)\, e^{i\Delta S\left[\xi,\bar{\xi};\lambda\right]}\delta\left(G^T\left(\xi-\bar{\xi}\right)\right)e^{-Q\left[\xi,\bar{\xi}\right]} \\ &= \frac{\mathcal{N}(\lambda)}{\mathcal{N}}D\left(\xi,\bar{\xi}\right)e^{i\frac{1}{\rho}\sum_{n=1}^{N}\frac{\lambda_n}{4!}\left(\xi_n^4-\bar{\xi}_n^4\right)}, \end{aligned} \tag{38}$$

where the constant $\mathcal{N}(\lambda)$ is chosen so that the decoherence functional is normalized, i.e.,

其中常数 $\mathcal{N}(\lambda)$ 的选取满足退相干泛函的归一化条件，即:

$$\int_{\mathbb{R}^{2N}} d^N\xi\, d^N\bar{\xi}\, D\left(\xi,\bar{\xi};\lambda\right) = 1. \tag{39}$$

It is not clear at first glance whether $\mathcal{N}(\lambda)$ is different than the constant $\mathcal{N}$ in the free case or whether $\mathcal{N}(\lambda)$ depends on λ at all, as indicated by the notation. As mentioned in [3], a further question is whether this new decoherence functional is positive semidefinite, as a general decoherence functional should be. Finally, we were motivated to write down this definition of the interacting decoherence functional by the continuum expression for the interacting decoherence functional in (13), but in the continuum theory, one can also (formally) express this interacting decoherence functional via the canonical framework (26). There is then the question of whether there exists such a canonical framework representation of the causal set $D\left(\xi,\overline{\xi};\lambda\right)$ and further whether this is analogous to the continuum expression in (26).

初看之下并不清楚 $\mathcal{N}(\lambda)$ 是否和自由情形中的常数 $\mathcal{N}$ 不同，也不清楚 $\mathcal{N}(\lambda)$ 是否真的如记号所示依赖于 λ。正如文献 [3] 中提到的，进一步的问题是: 这个新的退相干泛函是否满足一般退相干泛函应有的半正定性。最后，我们是通过式 (13) 中相互作用退相干泛函的连续统表达式得到了这个相互作用退相干泛函的定义，但在连续统理论中，也可以通过正则框架 (26)(形式上) 表示这个相互作用退相干泛函。因此还有问题: 因果集 $D\left(\xi,\overline{\xi};\lambda\right)$ 是否存在这样的正则框架表示，以及该表示是否和式 (26) 中的连续统表达式类似。

To finish this section, let us answer these questions. To begin, we rewrite the rhs of (38) using the free case definition (35):

在本节的最后，我们来回答这些问题。首先，我们利用自由情形的定义 (35) 重写式 (38) 的右侧:

$$D\left(\xi,\overline{\xi};\lambda\right)=\frac{\mathcal{N}(\lambda)}{\mathcal{N}}D\left(\xi,\overline{\xi}\right)e^{i\frac{1}{\rho}\sum_{n=1}^{N}\frac{\lambda_n}{4!}\left(\xi_n^4-\overline{\xi}_n^4\right)}$$

$$=\frac{\mathcal{N}(\lambda)}{\mathcal{N}}\left\langle\Omega\mid\delta\left(\phi_1-\overline{\xi}_1\right)\dots\delta\left(\phi_N-\overline{\xi}_N\right)\delta\left(\phi_N-\xi_N\right)\dots\right.$$

$$\left.\times\delta\left(\phi_1-\xi_1\right)|\Omega\right\rangle e^{i\frac{1}{\rho}\sum_{n=1}^{N}\frac{\lambda_n}{4!}\left(\xi_n^4-\overline{\xi}_n^4\right)}$$

$$=\frac{\mathcal{N}(\lambda)}{\mathcal{N}}\left\langle\Omega\right|\delta\left(\phi_1-\overline{\xi}_1\right)e^{-i\frac{\lambda_1}{\rho 4!}\overline{\xi}_1^4}\dots\delta\left(\phi_N-\overline{\xi}_N\right)e^{-i\frac{\lambda_N}{\rho 4!}\overline{\xi}_N^4}$$

$$\times\delta\left(\phi_N-\xi_N\right)e^{i\frac{\lambda_N}{\rho 4!}\xi_N^4}\dots\delta\left(\phi_1-\xi_1\right)e^{i\frac{\lambda_1}{\rho 4!}\xi_1^4}|\Omega\rangle.$$

(40)

Now, using the fact that, for any sufficiently well-behaved (By sufficiently well behaved, we mean the following: given the spectral measure $E_i(\cdot)$ associated with ϕ_i (which maps Borel subsets of $\mathbb{R}$ to projectors on the Fock space $\mathcal{F}$), and given any state $|\Psi\rangle\in\mathcal{F}$, the function $f:\mathbb{R}\to\mathbb{C}$ is square integrable against the real measure $\mu(\cdot)=\langle\Psi|E_i(\cdot)|\Psi\rangle$ (which maps Borel subsets of $\mathbb{R}$ to non-negative real numbers).) function $f:\mathbb{R}\to\mathbb{C}$, we have

现在，利用对任意充分良态的 (充分良态指: 给定与 ϕ_i 关联的谱测度 $E_i(\cdot)$ ，该谱测度将 $\mathbb{R}$ 的博雷尔子集映射到福克空间 $\mathcal{F}$ 上的投影，再给定任意态 $|\Psi\rangle\in\mathcal{F}$ ，函数 $f:\mathbb{R}\to\mathbb{C}$ 关于实测度 $\mu(\cdot)=\langle\Psi|E_i(\cdot)|\Psi\rangle$ 平方可积，其中实测度 $\mu(\cdot)=\langle\Psi|E_i(\cdot)|\Psi\rangle$ 将 $\mathbb{R}$ 的博雷尔子集映射到非负实数。) 函数 $f:\mathbb{R}\to\mathbb{C}$ 成立的性质，我们有

$$\int_{\mathbb{R}}da\delta\left(\phi_i-a\right)f(a)=f\left(\phi_i\right)=\int_{\mathbb{R}}da\delta\left(\phi_i-a\right)f\left(\phi_i\right),\tag{41}$$

our normalization condition then gives

我们的归一化条件于是给出

$$1=\int_{\mathbb{R}^{2N}} d^N\xi d^N\bar{\xi} D\left(\xi,\bar{\xi};\lambda\right)$$

$$=\frac{\mathcal{N}(\lambda)}{\mathcal{N}}\int_{\mathbb{R}^{2N}} d^N\xi d^N\bar{\xi}\left\langle\Omega\mid\delta\left(\phi_1-\bar{\xi}_1\right)e^{-i\frac{\lambda_1}{\rho 4!}\bar{\xi}_1^4}\ldots\delta\left(\phi_N-\bar{\xi}_N\right)e^{-i\frac{\lambda_N}{\rho 4!}\bar{\xi}_N^4}\right\rangle$$

$$\times\delta\left(\phi_N-\xi_N\right)e^{i\frac{\lambda_N}{\rho 4!}\xi_N^4}\ldots\delta\left(\phi_1-\xi_1\right)e^{i\frac{\lambda_1}{\rho 4!}\xi_1^4}|\Omega\rangle$$

$$=\frac{\mathcal{N}(\lambda)}{\mathcal{N}}\langle\Omega|\int_{\mathbb{R}} d\bar{\xi}_1\delta\left(\phi_1-\bar{\xi}_1\right)e^{-i\frac{\lambda_1}{\rho 4!}\bar{\xi}_1^4}\ldots\int_{\mathbb{R}} d\bar{\xi}_N\delta\left(\phi_N-\bar{\xi}_N\right)e^{-i\frac{\lambda_N}{\rho 4!}\bar{\xi}_N^4}$$

$$\times\int_{\mathbb{R}} d\xi_N\delta\left(\phi_N-\xi_N\right)e^{i\frac{\lambda_N}{\rho 4!}\xi_N^4}\ldots\int_{\mathbb{R}} d\xi_1\delta\left(\phi_1-\xi_1\right)e^{i\frac{\lambda_1}{\rho 4!}\xi_1^4}|\Omega$$

Ω

$$=\frac{\mathcal{N}(\lambda)}{\mathcal{N}}\langle\Omega\left|e^{-i\frac{\lambda_1}{\rho 4!}\phi_1^4}\ldots e^{-i\frac{\lambda_N}{\rho 4!}\phi_N^4}e^{i\frac{\lambda_N}{\rho 4!}\phi_N^4}\ldots e^{i\frac{\lambda_1}{\rho 4!}\phi_1^4}\right|\Omega\rangle. \tag{42}$$

The exponentials then cancel each other, one by one, leaving $1=\frac{\mathcal{N}(\lambda)}{\mathcal{N}}\langle\Omega\mid\Omega\rangle$. Since $\langle\Omega\mid\Omega\rangle=1$, we get $\mathcal{N}(\lambda)=\mathcal{N}$, which answers our normalization question.

指数项会逐一相互抵消，最终得到 $1=\frac{\mathcal{N}(\lambda)}{\mathcal{N}}\langle\Omega\mid\Omega\rangle$ 。由于 $\langle\Omega\mid\Omega\rangle=1$ ，我们得到 $\mathcal{N}(\lambda)=\mathcal{N}$ ，这就回答了归一化问题。

Regarding our question about a canonical framework representation of the causal set $D\left(\xi,\bar{\xi};\lambda\right)$, analogous to (26), Equations (41) and (40) tell us that we can write the interacting decoherence functional as

针对我们关于因果集 $D\left(\xi,\bar{\xi};\lambda\right)$ 的正则框架表示这一问题，类似式 (26)，式 (41) 和式 (40) 告诉我们可以将相互作用退相干泛函写为

$$D\left(\xi,\bar{\xi};\lambda\right)=\frac{\mathcal{N}(\lambda)}{\mathcal{N}}\left\langle\Omega\left|\delta\left(\phi_1-\bar{\xi}_1\right)e^{-i\frac{\lambda_1}{\rho 4}\bar{\xi}_1^4}\ldots\delta\left(\phi_N-\bar{\xi}_N\right)e^{-i\frac{\lambda_N}{\rho 4}\bar{\xi}_N^4}\right.\right\rangle$$

$$\times\delta\left(\phi_N-\xi_N\right)e^{i\frac{\lambda_N}{\rho 4!}\xi_N^4}\ldots\delta\left(\phi_1-\xi_1\right)e^{i\frac{\lambda_1}{\rho 4!}\xi_1^4}|\Omega\rangle$$

$$=\left\langle\Omega\left|\delta\left(\phi_1-\bar{\xi}_1\right)e^{-i\frac{\lambda_1}{\rho 4!}\phi_1^4}\ldots\delta\left(\phi_N-\bar{\xi}_N\right)e^{-i\frac{\lambda_N}{\rho 4!}\phi_N^4}\right|\right\rangle$$

$$\times\delta\left(\phi_N-\xi_N\right)e^{i\frac{\lambda_N}{\rho 4!}\phi_N^4}\ldots\delta\left(\phi_1-\xi_1\right)e^{i\frac{\lambda_1}{\rho 4!}\phi_1^4}|\Omega\rangle, \tag{43}$$

where we have also used our previous result that $\mathcal{N}(\lambda)=\mathcal{N}$. The last expression is immediately comparable to the formal continuum expression in (26). The ρ^{-1} factor in each exponent is analogous to the dV volume element in each exponent in (26).

其中我们也用到了之前得到的 $\mathcal{N}(\lambda) = \mathcal{N}$ 这一结论。该最终表达式可直接与式 (26) 中的形式连续表达式对比。每个指数中的 ρ^{-1} 因子，对应于式 (26) 中每个指数里的 dV 体积元。

The final question of whether $D\left(\xi, \overline{\xi}; \lambda\right)$ is positive semidefinite is more technically challenging to prove, but likely true nonetheless. Let us sketch an argument here.

最后一个问题，即证明 $D\left(\xi, \overline{\xi}; \lambda\right)$ 是半正定的，技术上更具挑战性，但该结论很可能成立。我们在此大致给出一个论证思路。

To phrase the positive semidefinite property, we need to introduce some concepts. The space of possible field configurations ξ is $\mathbb{R}^N$ - one real number for each causal set point. This is our sample space. The event algebra, $\mathfrak{A}$, over this sample space is then some σ -algebra on $\mathbb{R}^N$. A natural choice is the Borel σ -algebra, which is the smallest σ -algebra that contains the open sets. Each Borel subset $B \in \mathfrak{A}$ is called an event and corresponds to some subset of field configurations. For example, B could be the set of field configurations for which the value of ξ at point x is in the interval $[0, 1]$.

为了表述半正定性质，我们需要先引入一些概念。所有可能场构型构成的空间 ξ 是 $\mathbb{R}^N$ ——每个因果集点对应一个实数，这就是我们的样本空间。定义在该样本空间上的事件代数 $\mathfrak{A}$ 是 $\mathbb{R}^N$ 上的某个 σ σ 代数。一个自然的选择是博雷尔 σ σ 代数，它是包含开集的最小 σ σ 代数。每个博雷尔子集 $B \in \mathfrak{A}$ 都称为一个事件，对应某一类场构型的集合。例如，B 可以是满足 ξ 在点 x 处的值属于区间 $[0, 1]$ 的所有场构型的集合。

Now, for any pair of events $B, \bar{B} \in \mathfrak{A}$, we write

现在，对任意一对事件 $B, \bar{B} \in \mathfrak{A}$ ，我们记

$$D\left(B, \bar{B}; \lambda\right) = \int_B d^N\xi \int_{\bar{B}} d^N\overline{\xi} D\left(\xi, \overline{\xi}; \lambda\right), \tag{44}$$

though it is not clear at this point whether such integrals are defined for all Borel $B, \bar{B} \in \mathfrak{A}$. Assuming this is the case, for any finite set of events, $\mathfrak{B} = \{B_1, \ldots, B_r\} \subset \mathfrak{A}$, we define the event matrix $D_{ab} := D\left(B_b, B_a; \lambda\right)$, for $a, b = 1, \ldots, r$.

尽管目前尚不清楚这类积分是否对所有博雷尔 $B, \bar{B} \in \mathfrak{A}$ 都有定义。假设该条件成立，我们对任意有限事件集合 $\mathfrak{B} = \{B_1, \ldots, B_r\} \subset \mathfrak{A}$ ，在 $a, b = 1, \ldots, r$ 上定义事件矩阵 $D_{ab} := D\left(B_b, B_a; \lambda\right)$ 。

We say the decoherence functional is positive semidefinite if, for all finite sets of events, the corresponding event matrix is positive semidefinite, i.e., $\sum_{a,b=1}^{r} v_a^* D_{ab} v_b \geq 0$ for all $v \in \mathbb{C}^r$. This is also called strong positivity in [13].

我们称退相干泛函是半正定的，当且仅当对所有有限事件集合，对应的事件矩阵都是半正定的，即对任意 $v \in \mathbb{C}^r$ 都满足 $\sum_{a,b=1}^{r} v_a^* D_{ab} v_b \geq 0$ 。这在文献 [13] 中也被称为强正性。

For any finite set of events $\mathfrak{B} = \{B_1, \ldots, B_r\} \subset \mathfrak{A}$, and any $v \in \mathbb{C}^r$, we have

对任意有限事件集合 $\mathfrak{B} = \{B_1, \ldots, B_r\} \subset \mathfrak{A}$ 和任意 $v \in \mathbb{C}^r$ ，我们有

$$\sum_{a,b=1}^{r} v_a^* D_{ab} v_b = \sum_{a,b=1}^{r} v_a^* D\left(B_b, B_a; \lambda\right) v_b$$

$$= \sum_{a,b=1}^{r} v_a^* v_b \int_{B_b} d^N\xi \int_{B_a} d^N\bar{\xi} D\left(\xi, \bar{\xi}; \lambda\right)$$

$$= \left\langle \Omega \left| \sum_{a=1}^{r} v_a^* \int_{B_a} d^N\bar{\xi}\delta\left(\phi_1 - \bar{\xi}_1\right) e^{-i\frac{\lambda_1}{\rho 4!}\bar{\xi}_1^4} \ldots \delta\left(\phi_N - \bar{\xi}_N\right) e^{-i\frac{\lambda_N}{\rho 4!}\bar{\xi}_N^4} \right| \right\rangle$$

$$\times \sum_{b=1}^{r} v_b \int_{B_b} d^N\xi\delta\left(\phi_N - \xi_N\right) e^{i\frac{\lambda_N}{\rho 4!}\xi_N^4} \ldots \delta\left(\phi_1 - \xi_1\right) e^{i\frac{\lambda_1}{\rho 4!}\xi_1^4} |\Omega\rangle.$$

(45)

To properly complete this sketch of an argument, one needs to show that

要完整完成这个论证框架，需要证明

$$O\left(v, B\right) := \sum_{b=1}^{r} v_b \int_{B_b} d^N\xi\delta\left(\phi_N - \xi_N\right) e^{i\frac{\lambda_N}{\rho \cdot 1}\xi_N^4} \ldots \delta\left(\phi_1 - \xi_1\right) e^{i\frac{\lambda_1}{\rho \cdot 1}\xi_1^4}, \tag{46}$$

is a well-defined operator on the state $|\Omega\rangle$ for any finite set of events $\mathfrak{B} \subset \mathfrak{A}$ and any $v \in \mathbb{C}^r$. Assuming $O\left(v, B\right)$ is a well-defined operator on $|\Omega\rangle$, then $O\left(v, B\right)|\,\Omega\rangle$ is simply some state $|\Psi\rangle \in \mathcal{F}$, and the last line of (45) amounts to $\langle\Psi$ always non-negative.

对任意有限事件集合 $\mathfrak{B} \subset \mathfrak{A}$ 和任意 $v \in \mathbb{C}^r$ ，它都是态空间 $|\Omega\rangle$ 上一个良定义的算子。若假设 $O\left(v, B\right)$ 是 $|\Omega\rangle$, then $O\left(v, B\right)|\,\Omega\rangle$ 上良定义的算子，那么它本身就是某个非负态 $|\Psi\rangle \in \mathcal{F}$, and the last line of (45) amounts to $\langle\Psi \mid \Psi\rangle$, which is | ，恒为非负。

In fact, to show $O\left(v, B\right)$ is a well-defined operator on $|\Omega\rangle$, one needs to only show that $(*)\, O\left(B\right)$ is a well-defined operator on $|\Omega\rangle$ for any Borel subset $B \subseteq \mathbb{R}^N$, where

事实上，要证明 $O\left(v, B\right)$ 是 $|\Omega\rangle$ 上的良定义算子，只需证明对任意博雷尔子集 $B \subseteq \mathbb{R}^N$ ，$(*)\, O\left(B\right)$ 都是 $|\Omega\rangle$ 上的良定义算子，其中

$$O\left(B\right) := \int_{B} d^N\xi\delta\left(\phi_N - \xi_N\right) e^{i\frac{\lambda_N}{\rho 4!}\xi_N^4} \ldots \delta\left(\phi_1 - \xi_1\right) e^{i\frac{\lambda_1}{\rho 4!}\xi_1^4}. \tag{47}$$

If $(*)$ is the case, then any finite linear combination of operators of the form $O\left(B\right)$, as in (46), is still a well-defined operator.

若 $(*)$ 成立，则形如 $O\left(B\right)$ 的算子的任意有限线性组合 (如式 (46) 所示) 仍是良定义算子。

Actually proving (*) is technically involved, and we will not attempt it here. One suggested route for a proof is to start with Borel subsets $B \subseteq \mathbb{R}$ that are N -dimensional boxes, i.e., of the form $B = I_1 \times \ldots \times I_N$ for (potentially unbounded) intervals $I_1, \ldots, I_N \subseteq \mathbb{R}$. In this case,

对 (*) 的实际证明在技术上相当复杂，本文不展开讨论。一种可行的证明思路是从形如 N 维长方体的博雷尔子集 $B \subseteq \mathbb{R}$ 出发，即对 (可能无界的) 区间 $I_1, \ldots, I_N \subseteq \mathbb{R}$，子集形如 $B = I_1 \times \ldots \times I_N$。在这种情况下，

$$
\begin{aligned}
O(I_1 \times \ldots \times I_N) &= \int_{I_N} d\xi_N \delta(\phi_N - \xi_N) e^{i\frac{\lambda_N}{\rho 4!}\xi_N^4} \ldots \int_{I_1} d\xi_1 \delta(\phi_1 - \xi_1) e^{i\frac{\lambda_1}{\rho 4!}\xi_1^4} \\
&= E_N(I_N) e^{i\frac{\lambda_N}{\rho 4!}\phi_N^4} \ldots E_1(I_1) e^{i\frac{\lambda_1}{\rho 4!}\phi_1^4},
\end{aligned} \tag{48}
$$

where $E_n(\cdot)$ is the projection-valued measure associated with ϕ_n, which maps Borel subsets of $\mathbb{R}$ to projectors on the Fock space $\mathcal{F}$. The last line follows from functional calculus [14]. Since the last line is a finite product of projectors and unitaries of the form $e^{i\frac{\lambda_n}{\rho 4!}\phi_n^4}$, it is a well-defined operator.

其中 $E_n(\cdot)$ 是与 ϕ_n 关联的投影值测度，它将 $\mathbb{R}$ 的博雷尔子集映射为福克空间 $\mathcal{F}$ 上的投影。最后一行由泛函演算 [14] 得到。由于最后一行是投影与形如 $e^{i\frac{\lambda_n}{\rho 4!}\phi_n^4}$ 的酉算子的有限乘积，因此它是良定义算子。

For a general Borel subset $B \subseteq \mathbb{R}^N$, one can then imagine taking a limit of a disjoint union of boxes that tends to B in some sense and simultaneously taking the limit of the corresponding operators to furnish a well-defined $O(B)$. Such procedures are routine in measure theory and functional calculus [14] and will likely not cause any insurmountable roadblocks. Nevertheless, the fact we are dealing with operator-valued expressions like (47) should warrant some caution. We leave this calculation, and thus the proof of strong positivity, for future work.

对于一般的博雷尔子集 $B \subseteq \mathbb{R}^N$，我们可以取一系列长方体的不交并，使其在某种意义下趋近 B，同时对对应算子取极限，得到良定义的 $O(B)$。这类处理在测度论和泛函演算中是常规操作 [14]，大概率不会产生无法解决的问题。但我们仍需注意，本文处理的是式 (47) 这类算子值表达式。我们将这一计算以及强正性的证明留给未来的工作。

Interacting 2-Point Function

相互作用两点函数

To make contact with what is done in the continuum, let us focus on the interacting 2-point function $\left\langle \Omega \middle| \phi_x^{(H)} \phi_y^{(H)} \middle| \Omega \right\rangle$, for two causal set points $x, y \in C$. Let us further restrict ourselves to the causally ordered 2-point function, $\left\langle \Omega \middle| C\left\{\phi_x^{(H)} \phi_y^{(H)}\right\} \middle| \Omega \right\rangle$, as one can always recover the 2-point function from this if desired. Specifically, if $x \succcurlyeq y$ then $\left\langle \Omega \middle| \phi_x^{(H)} \phi_y^{(H)} \middle| \Omega \right\rangle = \left\langle \Omega \middle| C\left\{\phi_x^{(H)} \phi_y^{(H)}\right\} \middle| \Omega \right\rangle$, and if $x \preccurlyeq y$ then $\left\langle \Omega \middle| \phi_x^{(H)} \phi_y^{(H)} \middle| \Omega \right\rangle = \left\langle \Omega \middle| C\left\{\phi_x^{(H)} \phi_y^{(H)}\right\} \middle| \Omega \right\rangle^*$.

为了衔接连续谱中的现有研究，我们聚焦于两个因果集点 $x, y \in C$ 的相互作用两点函数 $\left\langle \Omega \middle| \phi_x^{(H)} \phi_y^{(H)} \middle| \Omega \right\rangle$。我们进一步将研究范围限定为因果序两点函数 $\left\langle \Omega \middle| C\left\{\phi_x^{(H)} \phi_y^{(H)}\right\} \middle| \Omega \right\rangle$，因为需要时总能由此还原出一般两点函数。具体而言，若 $x \succcurlyeq y$ 则 $\left\langle \Omega \middle| \phi_x^{(H)} \phi_y^{(H)} \middle| \Omega \right\rangle = \left\langle \Omega \middle| C\left\{\phi_x^{(H)} \phi_y^{(H)}\right\} \middle| \Omega \right\rangle$，若 $x \preccurlyeq y$ 则 $\left\langle \Omega \middle| \phi_x^{(H)} \phi_y^{(H)} \middle| \Omega \right\rangle = \left\langle \Omega \middle| C\left\{\phi_x^{(H)} \phi_y^{(H)}\right\} \middle| \Omega \right\rangle^*$.

For convenience, let us then assume that $x \npreceq y$, so that $C\left\{\phi_x^{(H)}\phi_y^{(H)}\right\} = \phi_x^{(H)}\phi_y^{(H)}$. Recalling our assumed natural labelling of C , we then know that, as labels, $x \geq y$.

为方便起见，我们接下来假设 $x \npreceq y$ ，由此可得 $C\left\{\phi_x^{(H)}\phi_y^{(H)}\right\} = \phi_x^{(H)}\phi_y^{(H)}$ 。回顾我们对 C 假定的自然标记法，我们可知作为标记，$x \geq y$ 。

To compute $\left\langle\Omega\left|\phi_x^{(H)}\phi_y^{(H)}\right|\Omega\right\rangle$ we integrate $\xi_x\xi_y$ against our interacting decoherence functional, $D\left(\xi,\bar{\xi};\lambda\right)$. Using (43) we find

为计算 $\left\langle\Omega\left|\phi_x^{(H)}\phi_y^{(H)}\right|\Omega\right\rangle$ ，我们将 $\xi_x\xi_y$ 对相互作用退相干泛函 $D\left(\xi,\bar{\xi};\lambda\right)$ 积分。利用式 (43) 我们得到

$$\left\langle\Omega\left|\phi_x^{(H)}\phi_y^{(H)}\right|\Omega\right\rangle = \int_{\mathbb{R}^{2N}} d^N\xi d^N\bar{\xi} D\left(\xi,\bar{\xi};\lambda\right)\xi_x\xi_y$$

$$= \left\langle\Omega\left|V_{x-1,\dots,1}^{\dagger}\phi_x V_{x-1,\dots,y}\phi_y V_{y-1,\dots,1}\right|\Omega\right\rangle, \tag{49}$$

where we have defined the unitary operator

其中我们定义了幺正算符

$$V_{b,b-1,\dots,a+1,a} := e^{i\frac{\lambda_b}{\rho 4!}\phi_b^4} e^{i\frac{\lambda_{b-1}}{\rho 4!}\phi_{b-1}^4} \dots e^{i\frac{\lambda_{a+1}}{\rho 4!}\phi_{a+1}^4} e^{i\frac{\lambda_a}{\rho 4!}\phi_a^4}, \tag{50}$$

for two labels $a < b$. By inserting $\mathbb{1} = V_{y-1,\dots,1}V_{y-1,\dots,1}^{\dagger}$, we can further rewrite the last line of (49) as

对应两个标记 $a < b$ 。代入 $\mathbb{1} = V_{y-1,\dots,1}V_{y-1,\dots,1}^{\dagger}$ 后，我们可进一步将 (49) 的最后一行改写为

$$\left\langle\Omega\left|\phi_x^{(H)}\phi_y^{(H)}\right|\Omega\right\rangle = \left\langle\Omega\left|V_{x-1,\dots,1}^{\dagger}\phi_x V_{x-1,\dots,y}\phi_y V_{y-1,\dots,1}\right|\Omega\right\rangle$$

$$= \langle\Omega|\, V_{x-1,\dots,1}^{\dagger}\phi_x V_{x-1,\dots,y}\left(V_{y-1,\dots,1}V_{y-1,\dots,1}^{\dagger}\right)\phi_y V_{y-1,.}$$

1

$$|\Omega\rangle$$

$$= \left\langle\Omega\left|V_{x-1,\dots,1}^{\dagger}\phi_x V_{x-1,\dots,1}V_{y-1,\dots,1}^{\dagger}\phi_y V_{y-1,\dots,1}\right|\Omega\right\rangle. \tag{51}$$

The last line reveals the unitary relationship between the interacting picture fields, ϕ_x , which carry the free dynamics, and the Heisenberg fields, $\phi_x^{(H)}$, which carry the full dynamics. Concretely, we have $\phi_x^{(H)} = V_{x-1,\dots,1}^{\dagger}\phi_x V_{x-1,\dots,1}$ for any $x \in C$.

最后一行揭示了相互作用绘景场 ϕ_x 与海森堡场 $\phi_x^{(H)}$ 之间的幺正关系: 前者承载自由动力学，后者承载全动力学。具体而言，对任意 $x \in C$ 都有 $\phi_x^{(H)} = V_{x-1,\dots,1}^{\dagger}\phi_x V_{x-1,\dots,1}$ 。

We can simplify this unitary relationship slightly. Since all of the exponentials, $e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$, in $V_{x-1,\dots,1}$ are in terms of fields ϕ_z for which the labels satisfy $z < x$, we know that the corresponding points $z \in C$ are either to the past of x or spacelike to x . If we further assume a natural labelling for which all of the points to the past of x have smaller labels than those that are spacelike to x , then the unitary splits into the product $V_{x-1,\ldots,1} = V_{x-1,\ldots,r+1}V_{r,\ldots,1}$, where r is the number of points to the past of x . Since all of the exponentials in $V_{x-1,\ldots,r+1}$ are in terms of fields which commute with ϕ_x , we know that $V^\dagger_{x-1,\ldots,r+1}\phi_x V_{x-1,\ldots,r+1} = \phi_x$, and hence $V^\dagger_{x-1,\ldots,1}\phi_x V_{x-1,\ldots,1}$ reduces to an expression involving on the unitaries $e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$ for which $z \preceq x$, i.e.,

我们可以略微简化这个幺正关系。由于 $V_{x-1,\ldots,1}$ 中所有指数项 $e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$ 都以场 ϕ_z 表示，且场的标签满足 $z < x$ ，因此我们可知对应点 $z \in C$ 要么在 x 的过去，要么与 x 类空。如果我们进一步假设一个自然标记，满足 x 过去的所有点的标签都小于与 x 类空点的标签，那么幺正算符可分解为乘积 $V_{x-1,\ldots,1} = V_{x-1,\ldots,r+1}V_{r,\ldots,1}$ ，其中 r 是 x 过去的点的数量。由于 $V_{x-1,\ldots,r+1}$ 中所有指数项都由对易 ϕ_x 的场表示，我们可知 $V^\dagger_{x-1,\ldots,r+1}\phi_x V_{x-1,\ldots,r+1} = \phi_x$ ，因此 $V^\dagger_{x-1,\ldots,1}\phi_x V_{x-1,\ldots,1}$ 可化简为仅包含满足 $z \preceq x$ 的幺正算符 $e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$ 的表达式，即:

$$\phi_x^{(H)} = V_x^\dagger \phi_x V_x \tag{52}$$

where V_x is defined to be the product of unitary operators $e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$ (for $z \preceq x$ and $z \neq x$), arranged in any order consistent with the causal ordering on the points z in the past of x .

其中 V_x 定义为幺正算符 $e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$ (对应 $z \preceq x$ 和 $z \neq x$) 的乘积，排列顺序与 x 过去点 z 上的因果序任意一致。

The finiteness of the causal set C means that, after expanding the exponentials in (52), $\phi_x^{(H)}$ amounts to a finite series in λ and the (interaction picture) fields ϕ_z for which $z \preceq x$. To see this, consider the adjoint action of some exponential $e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$ on ϕ_x , i.e., $e^{-i\frac{\lambda_z}{\rho 4!}\phi_z^4}\phi_x e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$. Using the formula

因果集 C 的有限性意味着, 对 (52) 式中的指数展开后, $\phi_x^{(H)}$ 可化为 λ 和满足 $z \preceq x$ 的 (相互作用绘景下) 场 ϕ_z 的有限级数。为说明这一点, 考虑某个指数项 $e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$ 对 ϕ_x 的伴随作用, 即 $e^{-i\frac{\lambda_z}{\rho 4!}\phi_z^4}\phi_x e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4}$ 。利用公式

$$e^{-B}Ae^{B} = \sum_{n=0}^{\infty}\frac{1}{n!}[A,B]_n, \tag{53}$$

where $[A,B]_0 = A$, and $[A,B]_{n+1} = \big[[A,B]_n, B\big]$, we find

其中 $[A,B]_0 = A$ ，且 $[A,B]_{n+1} = \big[[A,B]_n, B\big]$ ，我们得到

$$\begin{aligned} e^{-i\frac{\lambda_z}{\rho 4!}\phi_z^4}\phi_x e^{i\frac{\lambda_z}{\rho 4!}\phi_z^4} &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(i\frac{\lambda_z}{\rho 4!}\right)^n\left[\phi_x,\phi_z^4\right]_n \\ &= \phi_x + i\frac{\lambda_z}{\rho 4!}\left[\phi_x,\phi_z^4\right] \\ &= \phi_x - \frac{\lambda_z}{\rho 3!}\Delta_{xz}\phi_z^3 \end{aligned} \tag{54}$$

where line 2 follows as all higher commutators with ϕ_z^4 vanish. Now, if we then consider the adjoint action of $e^{i\frac{\lambda_w}{\rho 4!}\phi_w^4}$ on the last line, we get

其中第二行成立是因为所有与 ϕ_z^4 的更高阶对易子都为零。现在，如果我们接着考虑 $e^{i\frac{\lambda_w}{\rho 4!}\phi_w^4}$ 对最后一行结果的伴随作用，可得

$$
\begin{aligned}
& e^{-i\frac{\lambda_w}{\rho 4!}\phi_w^4}\phi_x e^{i\frac{\lambda_w}{\rho 4!}\phi_w^4} - \frac{\lambda_z}{\rho 3!}\Delta_{xz} e^{-i\frac{\lambda_w}{\rho 4!}\phi_w^4}\phi_z^3 e^{i\frac{\lambda_w}{\rho 4!}\phi_w^4} \\
&= e^{-i\frac{\lambda_w}{\rho 4!}\phi_w^4}\phi_x e^{i\frac{\lambda_w}{\rho 4!}\phi_w^4} - \frac{\lambda_z}{\rho 3!}\Delta_{xz}\left(e^{-i\frac{\lambda_w}{\rho 4!}\phi_w^4}\phi_z e^{i\frac{\lambda_w}{\rho 4!}\phi_w^4}\right)^3 \\
&= \phi_x - \frac{\lambda_w}{\rho 3!}\Delta_{xw}\phi_w^3 - \frac{\lambda_z}{\rho 3!}\Delta_{xz}\left(\phi_z - \frac{\lambda_w}{\rho 3!}\Delta_{zw}\phi_w^3\right)^3,
\end{aligned}
\tag{55}
$$

which makes it clear that each adjoint action amounts to replacing each field as $\phi_x \mapsto \phi_x - \frac{\lambda_y}{\rho 3!}\Delta_{xy}\phi_y^3$. Since C is finite, we only have finitely many adjoint actions, and thus $\phi_x^{(H)}$ is a finite series in λ and the (interaction picture) fields to the past of x .

由此可以明确，每个伴随作用相当于按照 $\phi_x \mapsto \phi_x - \frac{\lambda_y}{\rho 3!}\Delta_{xy}\phi_y^3$ 替换每个场。由于 C 是有限的，我们只有有限个伴随作用，因此 $\phi_x^{(H)}$ 是关于 λ 和 x 过去的 (相互作用绘景) 场的有限级数。

Going further, we can actually invert this relationship to write any interaction picture field ϕ_x in terms of Heisenberg picture fields $\phi_z^{(H)}$ for z to the past of x . This can be done in a recursive fashion starting at the minimal elements of C . A minimal element $x \in C$ has nothing to its past, and thus $\phi_x^{(H)} = \phi_x$. Moving on to elements x with only minimal elements to their past, we then know that $\phi_x^{(H)}$ is simply ϕ_x plus some cubic terms in any fields ϕ_y for which y is minimal and to the past of x . Such a ϕ_y can be written as $\phi_y^{(H)}$, since y is minimal. We can then express $\phi_x^{(H)} = \phi_x + O\left({\phi_y}^3\right) = \phi_x + O\left({\phi_y^{(H)}}^3\right)$, which can be rearranged to give ϕ_x purely in terms of Heisenberg picture fields. This recursive argument can then be continued until all interaction picture fields are expressed in terms of Heisenberg picture fields.

进一步来看，我们实际上可以对这个关系求逆，将任意相互作用绘景场 ϕ_x 用 x 过去的 z 的海森堡绘景场 $\phi_z^{(H)}$ 表示。这可以从 C 的极小元开始递归完成: 一个极小元 $x \in C$ 的过去没有任何元素，因此 $\phi_x^{(H)} = \phi_x$ 。接下来处理过去只有极小元的元素 x ，我们可知 $\phi_x^{(H)}$ 就是 ϕ_x 加上任意满足 y 是极小元且在 x 过去的场 ϕ_y 的三次项。这样的 ϕ_y 可以写成 $\phi_y^{(H)}$ ，因为 y 是极小元。接下来我们可以表示出 $\phi_x^{(H)} = \phi_x + O\left({\phi_y}^3\right) = \phi_x + O\left({\phi_y^{(H)}}^3\right)$ ，整理后就能得到仅用海森堡绘景场表示的 ϕ_x 。这个递归论证可以一直进行下去，直到所有相互作用绘景场都用海森堡绘景场表示出来。

With this relationship in hand, we can define the algebra of the interacting theory abstractly, i.e., without reference to a Hilbert space representation. Given any two Heisenberg fields $\phi_x^{(H)}$ and $\phi_y^{(H)}$, we can determine their commutator, $\left[\phi_x^{(H)}, \phi_y^{(H)}\right]$, purely in terms of other Heisenberg fields. To do this, we first expand each Heisenberg field in terms of the interaction picture fields. We then use the commutation relations for interaction picture fields, i.e., $[\phi_z, \phi_w] = i\Delta_{zw}\mathbb{1}$, to write $\left[\phi_x^{(H)}, \phi_y^{(H)}\right]$ as a finite polynomial in interaction picture fields. Finally, we use our inverse relationship to rewrite each such interaction picture field back in terms of Heisenberg fields. The result will likely be some complicated expression in terms of Heisenberg fields to the past of x and y , but it will be a finite series in λ and can be computed in principle. Together with the, already defined, action of † on polynomials in Heisenberg fields, these commutation relations written purely in terms of Heisenberg fields define the abstract *-algebra for the theory.

得到这个关系后，我们可以抽象地定义相互作用理论的代数，即不需要引用希尔伯特空间表示。给定任意两个海森堡场 $\phi_x^{(H)}$ 和 $\phi_y^{(H)}$ ，我们可以完全用其他海森堡场确定它们的对易子 $\left[\phi_x^{(H)}, \phi_y^{(H)}\right]$: 首先我们将每个海森堡场展开为相互作用绘景场的组合；接着利用相互作用绘景场的对易关系 (即 $[\phi_z, \phi_w] = i\Delta_{zw}\mathbb{1}$)，将 $\left[\phi_x^{(H)}, \phi_y^{(H)}\right]$ 写为相互作用绘景场的有限多项式；最后我们利用逆关系将每个相互作用绘景场换回海森堡场的形式。结果很可能是用 x 和 y 过去的海森堡场表示的复杂表达式，但它是关于 λ 的有限级数，原则上可以计算。结合已经定义好的 † 对海森堡场多项式的作用，这些完全用海森堡场表示的对易关系就定义了该理论的抽象 *-代数。

Before moving on, let us compare the 2-point function with the continuum expression in (2). Writing down the ratio of the two path integrals on rhs of (2) in the causal set case (Note we have double path integrals in the causal set case. If we integrate out the $\overline{\xi}$ field first, the resulting expression is more similar to (2).) gives

在继续之前，我们来将两点函数与 (2) 式中的连续谱表达式做比较。对因果集情形写出 (2) 式右侧两个路径积分的比值 (注意因果集情形下我们有双重路径积分，如果我们先对 $\overline{\xi}$ 场积分，得到的表达式会和 (2) 式更相似)，结果为

$$\frac{\int_{\mathbb{R}^{2N}} d^N\xi d^N\overline{\xi} D\left(\xi, \overline{\xi}; \lambda\right) \xi_x \xi_y}{\int_{\mathbb{R}^{2N}} d^N\xi d^N\overline{\xi} D\left(\xi, \overline{\xi}; \lambda\right)} = \left\langle \Omega \left| \phi_x^{(H)} \phi_y^{(H)} \right| \Omega \right\rangle, \tag{56}$$

since the denominator on the lhs is normalized to 1 . The state involved in the expectation value on the rhs is the free theory ground state $|\Omega\rangle$. In the continuum case, the expectation value is taken with the ground state of the interacting theory (denoted by $|0\rangle$ in (2)), and this is encoded in the path integral formulation by taking | the limit $T \to \infty(1 - i\varepsilon)$. There is also no analogue in the causal set case for this limit. We can make the causal set larger, i.e., $N \to \infty$, but this could only ever be comparable to $T \to \infty$ in the continuum. Let us leave this question for future investigations and continue on with our computation of $\left\langle \Omega \left| \phi_x^{(H)} \phi_y^{(H)} \right| \Omega \right\rangle$.

由于左侧分母已归一化为 1。右侧期望值对应的态是自由理论基态 $|\Omega\rangle$ 。在连续谱情形下，期望值由相互作用理论的基态计算 (记作 $|0\rangle$ in (2)), and this is encoded in the path integral formulation by taking |)，得到极限 $T \to \infty(1 - i\varepsilon)$ 。因果集情形下也不存在该极限的对应结构。我们可以增大因果集，即 $N \to \infty$ ，但这最多只能和连续谱情形下的 $T \to \infty$ 相近。我们将该问题留待未来研究，继续计算 $\left\langle \Omega \left| \phi_x^{(H)} \phi_y^{(H)} \right| \Omega \right\rangle$ 。

The Analogue of Feynman Diagrams

费曼图的类似物

Introduction

引言

Here, we work toward a diagram-based algorithm, akin to Feynman diagrams, for computing $\left\langle\Omega\left|\phi_x^{(H)}\phi_y^{(H)}\right|\Omega\right\rangle$ order by order in λ. The diagrams that arise in our case are almost identical to the usual ϕ^4 diagrams that appear in the perturbative expansion of the 2-point function (see Fig. 1 or Chapter 4 in [1] for examples), but are more complicated in that they contain two types of lines, or edges, between vertices. There are normal, or undirected, edges and directed edges indicated with arrows.

本文我们致力于推导一个类似费曼图的基于图的算法，用于按阶计算 λ 中的 $\left\langle\Omega\left|\phi_x^{(H)}\phi_y^{(H)}\right|\Omega\right\rangle$。我们这里得到的图，和两点函数微扰展开中常见的 ϕ^4 图 (例子可见图 1 或文献 [1] 第 4 章) 几乎完全一致，但复杂度更高: 顶点之间存在两种线即边，分别是常规无向边和箭头标记的有向边。

As a reminder, in continuum ϕ^4 theory [1] each vertex, z, in a diagram contributes an integral $(-i\lambda)\int_M d^dz$, and each edge between two vertices x and y contributes a factor of the Feynman propagator (time-ordered 2-point function) $G^F(x,y) = \left\langle 0\left|C\left\{\phi^{(H)}(x)\phi^{(H)}(y)\right\}\right|0\right\rangle$. Here, we have replaced the time ordering by a causal ordering, as the two are equivalent (since, if the two points are spacelike, spacelike commutativity allows us to reorder them however we please). Finally, one must also divide by the symmetry factor of the diagram.

提醒一下，在连续场 ϕ^4 理论 [1] 中，图里每个顶点 z 贡献一个积分 $(-i\lambda)\int_M d^dz$，两个顶点 x 和 y 之间的每条边贡献一个费曼传播子 (时序两点函数) 因子 $G^F(x,y) = \left\langle 0\left|C\left\{\phi^{(H)}(x)\phi^{(H)}(y)\right\}\right|0\right\rangle$。在本文中我们将时序替换为因果序，二者是等价的——因为如果两个点类空，类空对易性允许我们任意重排它们的顺序。最后还需要除以图的对称因子。

Returning to the causal set case, let us assume for simplicity that $x \not\prec y$ so that we can always pick a natural labelling for which $x \geq y$ (equality only if the two points are the same). We further assume a natural labelling for which all points not to the past of x and y have larger labels than x (and hence also y). Thus, for any point, z, strictly to the past of x or y, we know that its label satisfies $z < x$. Recalling (49), we have

回到因果集的情况，为简化分析我们假设 $x \not\prec y$，因此我们总能得到一个自然标记满足 $x \geq y$ (当且仅当两点重合时取等)。我们进一步假设，在该自然标记下，所有不属于 x 和 y 过去的点的标记都大于 x (因此也大于 y)。因此，对于严格位于 x 或 y 过去的任意点 z，其标记满足 $z < x$。回顾式 (49)，我们有

$$\begin{aligned}\phi_x^{(H)}\phi_y^{(H)} &= V^\dagger_{x-1,\dots,1}\phi_x V_{x-1,\dots y}\phi_y V_{y-1,\dots,1} \\ &= V^\dagger_{y-1,\dots,1}\left(V^\dagger_{x-1,\dots y}\phi_x V_{x-1,\dots y}\right)\phi_y V_{y-1,\dots,1}. \end{aligned} \tag{57}$$

For the term in brackets, we use the first line of (54) to get

对于括号内的项，我们利用 (54) 的第一行得到

$$\begin{aligned}V^\dagger_{x-1,\dots y}\phi_x V_{x-1,\dots y} = &\sum_{n_y,\dots,n_{x-1}=0}^{\infty}\frac{1}{n_y!\dots n_{x-1}!}\left(\frac{i}{\rho 4!}\right)^{n_y+\dots+n_{x-1}} \\ &\times\lambda_{x-1}^{n_{x-1}}\dots\lambda_y^{n_y}\left[\left[\dots\left[\phi_x,\phi_{x-1}^4\right]_{n_{x-1}},\dots\right]_{n_{y+1}},\phi_y^4\right]_{n_y}. \end{aligned} \tag{58}$$

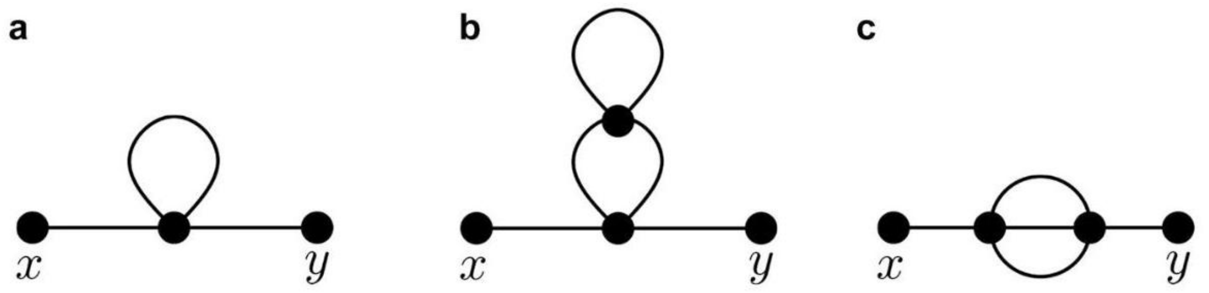


Fig. 1 Examples of connected diagrams in the 2-point function of a ϕ^4 theory

图 1 ϕ^4 理论中两点函数的连通图示例

It should then be clear that the last line of (57) can be written as

那么很明显，(57) 的最后一行可以写为

$$
\begin{aligned}
&V^\dagger_{y-1,\dots,1}\left(V^\dagger_{x-1,\dots y}\phi_x V_{x-1,\dots y}\right)\phi_y V_{y-1,\dots,1}\\
&=\sum_{n_1,\dots,n_{x-1}=0}^{\infty}\frac{1}{n_1!\dots n_{x-1}!}\left(\frac{i}{\rho 4!}\right)^{n_1+\dots+n_{x-1}}\lambda_{x-1}^{n_{x-1}}\dots\lambda_1^{n_1}\\
&\times\left[\left[\dots\left[\left[\left[\dots\left[\phi_x,\phi_{x-1}^4\right]_{n_{x-1}},\dots\right]_{n_{y+1}},\phi_y^4\right]_{n_y}\times\phi_y,\phi_{y-1}^4\right]_{n_{y-1}},\dots\right]_{n_2},\phi_1^4\right]_{n_1}.
\end{aligned}
$$

(59)

To tidy this up, we introduce the notation $[\leftarrow, A]$ to mean $B[\leftarrow, A] = [B, A]$, and we follow the convention that composition goes to the right, e.g., $A[\leftarrow, B][\leftarrow, C] = \ [A, B][-, C] = [[A, B], C]$. Now, we can rewrite the previous equation as

为了整理表达式，我们引入记号 $[\leftarrow, A]$ 表示 $B[\leftarrow, A] = [B, A]$，并遵循复合运算向右进行的约定，例如 $A[\leftarrow, B][\leftarrow, C] = \ [A, B][-, C] = [[A, B], C]$。现在我们可以将上式改写为

$$
\begin{aligned}
&V^\dagger_{y-1,\dots,1}\left(V^\dagger_{x-1,\dots y}\phi_x V_{x-1,\dots y}\right)\phi_y V_{y-1,\dots,1}\\
&=\sum_{n_1,\dots,n_{x-1}=0}^{\infty}\frac{1}{n_1!\dots n_{x-1}!}\left(\frac{i}{\rho 4!}\right)^{n_1+\dots+n_{x-1}}\lambda_{x-1}^{n_{x-1}}\dots\lambda_1^{n_1}\\
&\times L\left\{\phi_x\phi_y\left[\leftarrow,\phi_{x-1}^4\right]_{n_{x-1}}\dots\left[\leftarrow,\phi_1^4\right]_{n_1}\right\}
\end{aligned}
\tag{60}
$$

where we have also introduced the notation $L\{\cdot\}$ to denote the labelled order, which orders both field operators, ϕ_z, and commutators, $\left[\leftarrow, \phi_z^4\right]$, as if they were both the same abstract object, e.g., $L\left\{\phi_2\phi_4\left[\leftarrow,\phi_1^4\right]\left[\leftarrow,\phi_3^4\right]\right\} = \phi_4\left[\leftarrow,\phi_3^4\right]\phi_2\left[\leftarrow,\phi_1^4\right]$, which then evaluates to $\left[\left[\phi_4,\phi_3^4\right]\phi_2,\phi_1^4\right]$ under our composition convention. We can rewrite (60) as an expansion in orders of λ as

此处我们还引入记号 $L\{\cdot\}$ 表示标记序，该序将场算符 ϕ_z 和对易子 $[\leftarrow,\phi_z^4]$ 都视作同一抽象对象排序，例如 $L\{\phi_2\phi_4[\leftarrow,\phi_1^4][\leftarrow,\phi_3^4]\} = \phi_4[\leftarrow,\phi_3^4]\phi_2[\leftarrow,\phi_1^4]$，按照我们的组合约定，其结果为 $[[\phi_4,\phi_3^4]\phi_2,\phi_1^4]$。我们可将式 (60) 改写为按 λ 阶展开的形式

$$V^\dagger_{y-1,\ldots,1}\left(V^\dagger_{x-1,\ldots y}\phi_x V_{x-1,\ldots y}\right)\phi_y V_{y-1,\ldots,1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\rho 4!}\right)^n \sum_{z_1,\ldots,z_n=1}^{x-1}\lambda_{z_1}\ldots\lambda_{z_n}L\{\phi_x\phi_y[\leftarrow,\phi_{z_1}^4]\ldots[\leftarrow,\phi_{z_n}^4]\}, \tag{61}$$

which is the first step toward determining all possible Feynman diagrams that contribute toward $\left\langle\Omega\left|\phi_x^{(H)}\phi_y^{(H)}\right|\Omega\right\rangle$. More specifically, our diagrams must include at least two vertices - one for x and one for y - each with one free leg. Each term in the above sum then corresponds to a possible number of internal vertices to add, i.e., one vertex for each z_i $(i = 1, \ldots, n)$. Each internal vertex comes with four free legs, corresponding to the power of 4 that each ϕ_{z_i} appears with (see Fig. 2a, e.g.).

这是确定所有对 $\left\langle\Omega\left|\phi_x^{(H)}\phi_y^{(H)}\right|\Omega\right\rangle$ 有贡献的可能费曼图的第一步。更具体地说，我们的图必须至少包含两个顶点——一个对应 x ，一个对应 y ——每个顶点都有一条自由外腿。上述求和中的每一项对应于可能的内顶点添加数量，即每个 z_i $(i = 1, \ldots, n)$ 对应一个顶点。每个内顶点有四条自由外腿，对应每个 ϕ_{z_i} 自带的四次幂 (例如参见图 2a)。

Pre-diagrams

预图

Consider some fixed n and some particular values of $z_1, \ldots, z_n$ in the second sum $\sum_{z_1,\ldots,z_n=1}^{x-1}$ in (61). We now craft what we call pre-diagrams. To do this we first lay out the vertices $x, y, z_1, \ldots, z_n$ from left to right, going from largest to smallest. For example, if $n = 3$, and if the labels satisfy the chain of inequalities $x > z_2 > z_1 > y \geq z_3$, then we have the order of vertices shown in Fig. 2a. Laying out the vertices in this way takes into account the label ordering, $L\{\cdot\}$, in (61). Clearly as we go through the different values of $z_1, \ldots, z_n$ in the second sum $\sum_{z_1,\ldots,z_n=1}^{x-1}$, this order will differ. Let us stick with the example in Fig. 2 for now.

考虑 (61) 式第二个求和式 $\sum_{z_1,\ldots,z_n=1}^{x-1}$ 中某个固定的 n 和 $z_1, \ldots, z_n$ 的一组特定取值。现在我们来构造所谓的预图。首先我们将顶点 $x, y, z_1, \ldots, z_n$ 从左到右按标签从大到小排列。例如，若 $n = 3$，且标签满足不等式链 $x > z_2 > z_1 > y \geq z_3$， 则顶点顺序如图 2a 所示。这样排列顶点已经考虑了 (61) 式中的标签序 $L\{\cdot\}$。显然，当我们遍历第二个求和式 $\sum_{z_1,\ldots,z_n=1}^{x-1}$ 中 $z_1, \ldots, z_n$ 的不同取值时，该顺序会发生变化。我们暂时先以图 2 中的例子为例进行说明。

To make a pre-diagram, we then work through the internal vertices - the z_i ’ s - moving from left to right, i.e., in our example, we do z_2 first (Fig.2b), then z_1 (Fig.2c), and then z_3 (Fig.2d). For each z_i , we join 1 to 4 of its legs to any of the free legs coming from the vertices to the left of the given z_i (as in Fig. 2). The resulting edges are all directed to the left, as indicated by the arrows on the lines. Once we have worked through all the internal vertices in order, we call the result a pre-diagram - so-called because not all the legs are paired up, e.g., Fig. 2d.

要构造预图，我们需要从左到右依次处理内部顶点 (即 z_i)，在我们的例子中就是先处理 z_2 (图 2b)，再处理 z_1 (图 2c)，最后处理 z_3 (图 2d)。对每个 z_i ，我们将它的 1 到 4 条腿连接到该 z_i 左侧顶点伸出的任意自由腿上 (如图 2 所示)。生成的所有边都指向左方，如图中箭头所示。按顺序处理完所有内部顶点后，得到的结果就是预图——之所以叫预图，是因为并非所有腿都完成了配对，例如图 2d。

Focusing on a given z_i in this process, the act of joining up some non-zero number of its legs encodes the action of $\left[\leftarrow, \phi_{z_i}^4\right]$ in the expression $\phi_x\left[\leftarrow, \phi_{z_2}^4\right]\ \left[\leftarrow, \phi_{z_1}^4\right]\phi_y\left[\leftarrow, \phi_{z_3}^4\right]$ (note how the order of these operations matches that in Fig. 2). To see this, consider some points whose labels are ordered as $a_n \geq \ldots \geq a_1 \geq a_0$. It is not too hard to show the identity

在该过程中，对任意给定的 z_i ，连接它的若干条非零数量腿的操作对应了表达式 $\phi_x\left[\leftarrow, \phi_{z_2}^4\right]\ \left[\leftarrow, \phi_{z_1}^4\right]\phi_y\left[\leftarrow, \phi_{z_3}^4\right]$ 中 $\left[\leftarrow, \phi_{z_i}^4\right]$ 的作用 (注意这些操作的顺序与图 2 中的顺序一致)。我们来考虑一组标签满足顺序 $a_n \geq \ldots \geq a_1 \geq a_0$ 的点，不难证明以下恒等式

$$
\begin{aligned}
&\left[\phi_{a_n}\cdots\phi_{a_1}, \phi_{a_0}^4\right] \\
&= \left(\sum_{r=1}^{n} i\Delta_{a_r a_0}\phi_{a_n}\ldots(\mathbb{1})_{a_r}\ldots\phi_{a_1}\right)4\phi_{a_0}^3 \\
&+\left(\sum_{r>r'} i\Delta_{a_r a_0} i\Delta_{a_{r'} a_0}\phi_{a_n}\ldots(\mathbb{1})_{a_r}\ldots(\mathbb{1})_{a_{r'}}\ldots\phi_{a_1}\right)(-4.3)\,\phi_{a_0}^2 \\
&+\left(\sum_{r>r'>r''} i\Delta_{a_r a_0} i\Delta_{a_{r'} a_0} i\Delta_{a_{r''} a_0}\phi_{a_n}\ldots(\mathbb{1})_{a_r}\ldots(\mathbb{1})_{a_{r'}}\ldots(\mathbb{1})_{a_{r''}}\ldots\phi_{a_1}\right)4.3.2\phi_{a_0} \\
&+\left(\sum_{r>r'>r''>r'''} i\Delta_{a_r a_0} i\Delta_{a_{r'} a_0} i\Delta_{a_{r''} a_0} i\Delta_{a_{r''} a_0}\phi_{a_n}\ldots(\mathbb{1})_{a_r}\ldots(\mathbb{1})_{a_{r'}}\ldots(\mathbb{1})_{a_{r''}}\ldots(\mathbb{1})_{a_{r'''}}\ldots\phi_{a_1}\right) \\
&\times(-4.3.2.1)\ ,
\end{aligned}
\tag{62}
$$

where $(X)_{a_r}$ means we insert the operator X instead of ϕ_{a_r} in the r ’ th position in the product $\phi_{a_n}\ldots\phi_{a_1}$. Note that all of the products of fields on the rhs are label ordered. It is also clear that lines 1 to 4 on the rhs can be interpreted as joining 1 to 4 of a_0 ’ s four free legs with the free legs corresponding to each a_i (for $i = 1, \ldots, n$).

其中 $(X)_{a_r}$ 表示我们在乘积 $\phi_{a_n}\ldots\phi_{a_1}$ 的第 r 个位置插入算符 X 而非 ϕ_{a_r} 。注意右侧所有场乘积都符合标签序。显然，右侧第 1 至 4 行可以解释为: 将 a_0 的四条自由腿中的 1 至 4 条，与对应每个 a_i (对于 $i = 1, \ldots, n$) 的自由腿连接。

$x\ z_2\ z_1\ y\ z_3$
$X = \phi_x\left[\leftarrow, \phi_{z_2}^4\right]\left[\leftarrow, \phi_{z_1}^4\right]\phi_y\left[\leftarrow, \phi_{z_3}^4\right]$
(a) Initial setup with the vertices in

(a) 顶点初始排列

order.

完成。

$x\ z_2\ z_1\ y\ z_3$

$X = \left(iG_{xz_2}(4)_2\phi_{z_2}^3\right)\left[\leftarrow,\phi_{z_1}^4\right]\phi_y\left[\leftarrow,\phi_{z_3}^4\right]$

(b) Factor of $(4)_2$ ：4 ways to pick one

(b) $(4)_2$ ：4 因子对应从 $(4)_2$ ：4 的四条自由腿中选一条

of z_2 ’ s four free legs to join with x .

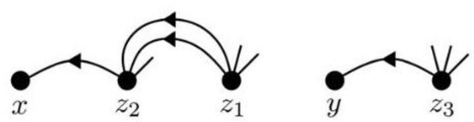
将 z_2 的四条自由外腿中的一条与 x 连接。

$z_2\ z_1\ y\ z_3$

$X = \left(iG_{xz_2}(4)_2\left(iG_{z_2z_1}\right)^2(-4.3)_1(3)_2\phi_{z_2}\phi_{z_1}^2 + \ldots\right)\phi_y\left[\leftarrow,\phi_{z_3}^4\right]$

(c) Factor of $(-4.3)_1$ ： 4 ways to pick the first, then 3 ways to pick the second of z_1 ’ s free legs to join with z_2 . Factor of $(3)_2$: $3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ ways to pick two of z_2 ’ s three free legs to join with the two chosen legs from z_1 .

(c) 有 $(-4.3)_1$ ：4 种因子方式选择第一个自由腿，再由 3 种方式选择 z_1 的第二个自由腿与 z_2 连接。因子 $(3)_2$: 有 $3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ 种方式从 z_2 的三个自由腿中选出两个，与来自 z_1 的两个选定腿连接。



$X = iG_{xz_2}(4)_2\left(iG_{z_2z_1}\right)^2(-4.3)_1(3)_2 iG_{yz_3}(4)_3\phi_{z_2}\phi_{z_1}^2\phi_y\phi_{z_3}^3 + ..$

(d) Factor of $(4)_3$ ：4 ways to pick one of z_3 ’ s four free legs to join

(d) 有 $(4)_3$ ：4 种因子方式从 z_3 的四个自由腿中选出一个进行连接

with y .

到 y 。

z_1

$x\ z_2\ z_1\ y\ z_3\ x\ z_2\ z_3\ y$

$\langle X\rangle = iG_{xz_2}(4)_2\left(iG_{z_2z_1}\right)^2(-4.3)_1(3)_2 iG_{yz_3}(4)_3(3)_{23}G^F_{z_2z_3}(2)_{13}G^F_{z_1z_3} + \ldots$

(e) Factor of $(3)_{23}$: 3 ways to pick one of z_3 ’ s three remaining legs to join with z_2 . Factor of $(2)_{13}$: 2 ways to pair up the remaining two legs from z_1 and z_3 . The rhs shows the same labelled diagram but arranged more like a Feynman diagram in the continuum theory.

(e) 因子 $(3)_{23}$: 有 3 种方式从 z_3 剩余的三个腿中选出一个与 z_2 连接。因子 $(2)_{13}$: 有 2 种方式将来自 z_1 和 z_3 的剩余两个腿配对。右侧展示了同一个标记图，但排列更接近连续谱理论中的费曼图。

Fig. 2 Step-by-step construction of a pre-diagram ((a) to (d)) and then a labelled diagram in (e). Underneath each graph, we have kept track of the relevant term in $X = \phi_x\left[\leftarrow, \phi_{z_2}^4\right]\left[\leftarrow, \phi_{z_1}^4\right]\ \phi_y\left[\leftarrow, \phi_{z_3}^3\right]$ and $\langle X\rangle \equiv \langle\Omega\,|X|\,\Omega\rangle$ in (e). The subscripts on any numerical factors, e.g., the " 1 " in $(-4.3)_1$, indicate the vertex, or vertices, that contributed to the given factors. The ellipses in the expressions for X and $\langle X\rangle$ denote all other terms comprising X and $\langle X\rangle$ that do not correspond to the given sub-figure

图 2 从 (a) 到 (d) 逐步构建预图，(e) 为标记图。在每个图下方，我们记录了 (e) 中 $X = \phi_x\left[\leftarrow, \phi_{z_2}^4\right]\left[\leftarrow, \phi_{z_1}^4\right]\ \phi_y\left[\leftarrow, \phi_{z_3}^3\right]$ 和 $\langle X\rangle \equiv \langle\Omega\,|X|\,\Omega\rangle$ 的相关项。所有数值因子的下标，例如 $(-4.3)_1$ 中的“1”，标记对该因子有贡献的一个或多个顶点。X 和 $\langle X\rangle$ 表达式中的省略号代表构成 X 和 $\langle X\rangle$ 的所有其他不对应本分图的项。

Each such join comes with a factor of $i\Delta$, which, because the indices of Δ are always label ordered, can actually be replaced by iG , where G is the retarded Green function (this explains the factors of iG in Fig. 2). We also get a factor of -1 if we join up an even number of a_0 ’ s legs (as in Fig. 2d). The factors of 4,3, and 2 come from the number of different choices we have when picking from a_0 ’ s legs. Thus, when constructing a pre-diagram, it helps to think of any free legs as being labelled.

每次这样的连接都带有因子 $i\Delta$ ，由于 Δ 的指标始终按标记排序，它实际上可以替换为 iG ，其中 G 是推迟格林函数 (这解释了图 2 中 iG 因子的来源)。如果我们连接偶数个 a_0 的腿 (如图 2d 所示)，我们还会得到因子-1。4、3 和 2 这些因子来源于选择 a_0 的腿时的不同选择数量。因此，在构建预图时，将所有自由腿视为已标记会更方便。

Back to our example expression $\phi_x\left[\leftarrow, \phi_{z_2}^4\right]\left[\leftarrow, \phi_{z_1}^4\right]\phi_y\left[\leftarrow, \phi_{z_3}^4\right]$. We can now express this as a sum of all possible pre-diagrams constructed from the vertices x, z_2, z_1, y, z_3 , in that order. For a given pre-diagram, such as that in Fig. 2d, we can read off the corresponding operator expression as follows. For every directed edge from a vertex a to b , we get a factor iG_{ba} . If a vertex has p directed edges coming out of it (where $p = 1, \ldots, 4$), then we get a factor of $(-1)^{p-1}$. For each internal vertex, we get a factor of $4!/\,(4-q)!$, where q is the number of ingoing plus outgoing directed edges for the vertex. If between two vertices there are s directed edges, we get a factor of $1/s\,!$. Finally, for each free leg coming from a vertex a , we get a field operator ϕ_a . All the field operators must be ordered according to the order of the vertices. Following these rules for the pre-diagram in Fig. 2d gives

回到我们的示例表达式 $\phi_x\left[\leftarrow, \phi_{z_2}^4\right]\left[\leftarrow, \phi_{z_1}^4\right]\phi_y\left[\leftarrow, \phi_{z_3}^4\right]$ 。现在我们可以将其表示为由顶点 x, z_2, z_1, y, z_3 按该顺序构造的所有可能预图的和。对于给定的预图，例如图 2d 中的预图，我们可以按如下方式读出对应的算符表达式: 从顶点 a 指向 b 的每条有向边，都对应一个因子 iG_{ba} 。如果一个顶点有 p 条出向有向边 (其中 $p = 1, \ldots, 4$)，那么我们得到一个因子 $(-1)^{p-1}$ 。对于每个内部顶点，我们得到一个因子 $4!/\,(4-q)!$ ，其中 q 是该顶点入向加出向有向边的总数。如果两个顶点之间有 s 条有向边，我们得到一个因子 $1/\ 1/s\,!$。最后，对于顶点 a 引出的每个自由支，我们得到一个场算符 ϕ_a 。所有场算符必须按顶点的顺序排列。对图 2d 中的预图应用这些规则可得

$$-\,(4.3.2)\,(4.3)\,(4)\,(1/2!)\,iG_{xz_2}\big(iG_{z_2z_1}\big)^2 iG_{yz_3}\phi_{z_2}\phi_{z_1}^2\phi_{z_3}^3\,, \tag{63}$$

where (-1) comes from the two directed legs leaving z_1 ,(4.3.2) comes from the 3 ingoing plus outgoing directed legs from z_2 ,(4.3) from the 2 outgoing directed legs from z_1 ,(4) from the 1 outgoing leg from z_1 , and (1/2!) from the 2 directed legs between z_1 and z_2 . The powers of the final fields represent how many free legs remain for each vertex.

其中 (-1) 来自离开 z_1 的两条有向支，(4.3.2) 来自 z_2 的 3 条入向加出向有向边，(4.3) 来自 z_1 的 2 条出向有向边，(4) 来自 z_1 的 1 条出向支，(1/2!) 来自 z_1 和 z_2 之间的 2 条有向边。最终场的幂次表示每个顶点剩余的自由支数量。

Labelled Diagrams

标记图

In the end, we are interested in computing the expectation value of $\phi_x^{(H)}\phi_y^{(H)}$ using the state $|\Omega\rangle$. Thus, we must take the expectation value of the operator expressions arising from any pre-diagrams. The Gaussian nature of $|\Omega\rangle$ means that, when acting on the product of fields $\phi_{z_2}\phi_{z_1}^2\phi_{z_3}^3 = \phi_{z_2}\phi_{z_1}\phi_{z_1}\phi_{z_3}\phi_{z_3}\phi_{z_3}$, for example, we get a sum over all the possible ways to pair up the fields into free 2-point functions. Some of these pairings result in the same expression in terms of 2-point functions, which introduces numerical factors into the result. For our example we get

最终，我们感兴趣的是利用态 $|\Omega\rangle$ 计算 $\phi_x^{(H)}\phi_y^{(H)}$ 的期望值。因此，我们必须对任意前图得到的算符表达式取期望值。$|\Omega\rangle$ 的高斯性质意味着，例如作用在场的乘积 $\phi_{z_2}\phi_{z_1}^2\phi_{z_3}^3 = \phi_{z_2}\phi_{z_1}\phi_{z_1}\phi_{z_3}\phi_{z_3}\phi_{z_3}$ 上时，我们会得到对所有可能配对方式的求和，即将所有场配成对得到自由两点函数。其中部分配对会得到形式完全相同的两点函数表达式，这就给结果引入了数值因子。对我们的例子而言可得

$$\langle\Omega\,|\phi_{z_2}\phi_{z_1}\phi_{z_1}\phi_{z_3}\phi_{z_3}\phi_{z_3}|\,\Omega\rangle = 6W_{21}W_{13}W_{33} + 3W_{23}W_{11}W_{33} + 6W_{23}W_{13}{}^2, \tag{64}$$

where the numerical factors count the possible ways to pair up the fields to get the associated 2-point functions. Note that the indices of the free 2-point functions are all ordered in a way that is consistent with the ordering of the fields in the original product. Since our fields were originally ordered based on their natural labelling, this means that any 2-point functions that appear can be written as causally ordered 2-point functions, or equivalently, Feynman propagators, $G_{ab}^F := \langle\Omega\,|C\{\phi_a\phi_b\}|\,\Omega\rangle$. This explains the factors of G^F in Fig. 2e.

其中数值因子统计了将场配成对得到对应两点函数的所有可能方式。注意,自由两点函数的所有指标的排序方式, 都与原乘积中各场的排序一致。由于我们的场最初是按其自然标记排序的, 这意味着所有出现的两点函数都可以写为因果序两点函数，等价地也就是费曼传播子 $G_{ab}^F := \langle\Omega\,|C\{\phi_a\phi_b\}|\,\Omega\rangle$ 。这就解释了图 2e 中 G^F 因子的来源。

The different terms on the rhs of (64) correspond to all the possible ways to pair up the remaining legs in our pre-diagram in Fig. 2d. The 3rd term on the rhs of (64) corresponds to the pairing shown in Fig. 2e. Pairing up the free legs of a pre-diagram furnishes a labelled diagram - so-called because its vertices are labelled z_1, z_2 , and so on. In a labelled diagram, every internal vertex, z_i , will have at least one route via directed edges to get to either x or y . Furthermore, every such route one can take (as there may be multiple choices of directed edges one can take from a given internal vertex) leads to either x or y (there are no closed loops of directed edges). The factors for a given labelled diagram are the same as above but with a factor of G^F_{ab} for every undirected edge between any two vertices a and b . There may also be a numerical factor if the same labelled diagram can arise in multiple ways from the same pre-diagram. The free legs of the pre-diagram in Fig. 2d can be paired in six different ways to get the labelled diagram in Fig. 2e - three ways to pick one of z_3 ’ s three free legs to pair with z_2 and then two ways to pair the remaining legs from z_1 and z_3 .

(64) 式右侧的不同项对应于图 2d 中前图剩余外腿的所有可能配对方式。(64) 式右侧第三项对应图 2e 所示的配对。将前图的自由外腿配成对就得到了标记图——之所以这么叫是因为它的顶点都被标记 z_1, z_2 ，依此类推。在标记图中，每个内顶点 z_i 至少存在一条沿有向边到达 x 或 y 的路径。此外，从任意给定内顶点出发的所有路径 (因为可能存在多条可选的有向边) 最终都到达 x 或 y (不存在有向边构成的闭合圈)。一个给定标记图的因子与上述规则一致，但任意两个顶点 a 和 b 之间的每条无向边都额外带一个因子 G^F_{ab} 。如果同一个标记图可以从同一个前图通过多种方式得到，还会额外带有一个数值因子。图 2d 中前图的自由外腿共有六种不同配对方式得到图 2e 的标记图——其中有三种方式选择 z_3 的三个自由外腿之一与 z_2 配对，之后剩下的来自 z_1 和 z_3 的外腿还有两种配对方式。

To summarize, we now have

总结一下，我们现在得到

$$\left\langle \Omega \left| \phi_x^{(H)} \phi_y^{(H)} \right| \Omega \right\rangle = \sum_{n=0} \frac{1}{n!} \left(\frac{i}{\rho 4!} \right)^n \sum_{z_1, \ldots, z_n = 1}^{x-1}$$

$$\times \lambda_{z_1} \ldots \lambda_{z_n} \left[\begin{array}{c} \text{all possible labelled diagrams} \\ \text{given the order of } z_i \text{ vertices} \end{array} \right], \tag{65}$$

where by ”all possible labelled diagrams ...,” we mean all those constructed by putting the vertices in their label order and following the procedure outlined above, i.e., first constructing the pre-diagram by forming directed edges, going one vertex at a time from left to right, and then joining up remaining legs as undirected edges. For different values of $z_1, \ldots, z_n$ in the sum $\sum\limits_{z_1, \ldots, z_n = 1}^{x-1}$, the possible labelled diagrams that can arise differ. Each labelled diagram contributes factors of iG for any directed edges, factors of the Feynman propagator G^F for any undirected edges, some numerical factors, and potentially a factor of -1 .

其中“所有可能的标记图……”指的是所有按如下方式构造得到的图: 将顶点按标记顺序排列，遵循上述步骤，即先构造前图，从左到右依次处理每个顶点，构建有向边，再将剩余外腿连接为无向边。求和式 $\sum\limits_{z_1, \ldots, z_n = 1}^{x-1}$ 中 $z_1, \ldots, z_n$ 取不同值时，得到的可能标记图也不同。每个标记图中，每条有向边贡献因子 iG ，每条无向边贡献费曼传播子因子 G^F ，此外还有若干数值因子，还可能带有一个因子-1。

Consider the labelled diagram in Fig. 2e which contributes the term shown under the graphs in Fig. 2e. We can make this look more like a typical diagram from the continuum ϕ^4 theory by rearranging the vertices, as we do on the rhs of Fig. 2e. We note, however, that our labelled diagrams differ from those in the continuum in that ours have both directed and undirected edges, and, at this stage, they have labelled vertices. We address the latter point now.

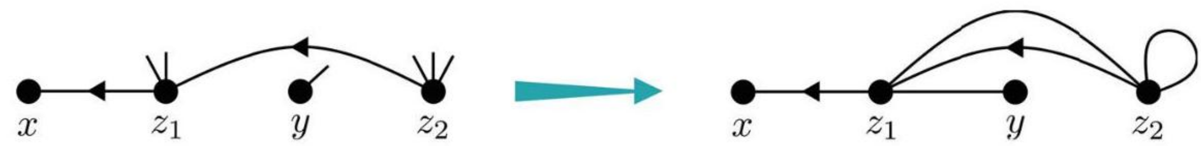
考虑图 2e 中的标号图，它贡献了图 2e 中曲线下方所示的项。通过重新排列顶点，我们可以让它看起来更像典型的连续统 ϕ^4 理论图，正如图 2e 右侧所示。但需要注意的是，我们的标号图与连续统中的标号图不同: 我们的图同时包含有向边和无向边，且目前所有顶点都是已标号的。下面我们就来讨论顶点标号这一点。

Unlabelled Diagrams

无标号图

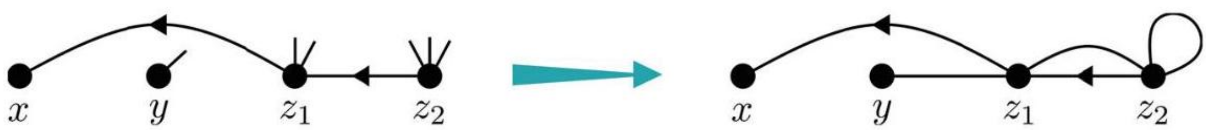
Consider the labelled diagram in Fig. 3c. This arises as a possible labelled diagram when the values of z_1 and z_2 in the sum $\sum_{z_1,z_2=1}^{x-1}$ are ordered as $x \geq z_1 \geq y \geq z_2$ (Fig. 3a), or as $x \geq y > z_1 \geq z_2$ (Fig. 3b). Thus, the ordering between z_1 and y does not matter, which one can infer from the undirected edge between z_1 and y in the labelled diagram (the rhs of Fig. 3a and b, or Fig. 3c).

考虑图 3c 中的标号图。当求和式 $\sum_{z_1,z_2=1}^{x-1}$ 中的 z_1 和 z_2 取值排序为 $x \geq z_1 \geq y \geq z_2$ (图 3a)，或排序为 $x \geq y > z_1 \geq z_2$ (图 3b) 时，就会得到这个可能的标号图。因此 z_1 和 y 之间的排序无关紧要，这一点可以从标号图 (图 3a 和 3b 的右侧，即图 3c) 中 z_1 与 y 之间的无向边推得。



(a) Pre-diagram to labelled diagram with starting order $x \geq z_1 \geq y \geq z_2$.

(a) 起始顺序为 $x \geq z_1 \geq y \geq z_2$ 的标号图的预图。



(b) Pre-diagram to labelled diagram with starting order $x \geq y > z_1 \geq z_2$.

(b) 起始顺序为 $x \geq y > z_1 \geq z_2$ 的标号图的预图。

invalid

无效

z_2

$x\ z_1$ Y x Y $z_2\ z_1$

$x\ z_1$ Y x Y $z_2\ z_1$

(c) Resulting (d) Invalid pre-diagram for

(c) 得到的标号图 (d) 无效预图，对应

labelled diagram starting order $z_2 > z_1$.

起始顺序为 $z_2 > z_1$ 的标号图。

Fig. 3 Example of how different starting orders can give the same pre-diagram and hence the same labelled diagram. (c) shows the labelled diagram that results from both (a) and (b). (d) is an invalid pre-diagram as a directed edge goes to the right

图 3 不同起始顺序如何得到相同预图、进而得到相同标号图的示例。(c) 展示了由 (a) 和 (b) 共同得到的标号图。(d) 是无效预图，因为有一条有向边指向右侧

If we change the ordering between any vertices connected by a directed edge, however, e.g., if we consider values in the sum for which $z_2 > z_1$, then this labelled diagram does not arise (see the invalid pre-diagram in Fig. 3d). It is more convenient, however, to think of it as being there, but since it comes with a factor of $G_{z_1 z_2}$ (which vanishes for $z_2 > z_1$), its contribution vanishes.

但如果我们改变任意一条有向边连接的顶点之间的顺序，例如，若我们考虑求和中满足 $z_2 > z_1$ 的取值，就不会得到这个标号图 (参见图 3d 中的无效预图)。不过更方便的处理方式是认为它依然存在，但由于它带有因子 $G_{z_1 z_2}$ (该因子在 $z_2 > z_1$ 时为零)，因此它的贡献为零。

In general, then, we can think of every $(n+2)$ -vertex labelled diagram as appearing in every term of the sum $\sum_{z_1,\ldots,z_n=1}^{x-1}$, and only for those values of $z_1, \ldots, z_n$ which match the ordering implied by the directed edges of the labelled diagram does the given diagram contribute something non-trivial.

因此一般来说，我们可以认为每个 $(n+2)$ 顶点标号图都出现在求和式 $\sum_{z_1,\ldots,z_n=1}^{x-1}$ 的每一项中，只有当 $z_1, \ldots, z_n$ 的取值符合标号图有向边隐含的排序关系时，该图才会给出非平凡贡献。

For any labelled diagram, we also get diagrams corresponding to permutations of its vertex labels, e.g., swapping z_1 and z_2 in Fig. 3c or z_1, z_2 and z_3 in Fig. 2e. This suggests that we group together all labelled diagrams that are related via a perturbation of their vertices into one unlabelled diagram. This takes into account the factor of $1/n!$ in (65). With this, we can also "absorb" the sum $\left(\frac{i}{\rho}\right)^n \sum_{z_1,\ldots,z_n=1}^{x-1}$ into these new unlabelled diagrams by letting each vertex, z , of the unlabelled diagram contribute a sum $\frac{i}{\rho} \sum_{z=1}^{x-1}$.

对于任意标号图, 我们还可以得到对应其顶点标签置换的图, 例如交换图 3c 中的 z_1 和 z_2 , 或交换图 2e 中的 z_1, z_2 和 z_3 。这提示我们将所有可通过顶点置换相互转化的标号图归为一类，合并为一个无标号图。这就将 (65) 式中的 $1/n\,!$ 因子考虑在内了。通过这种方式，我们还可以将求和 $\left(\frac{i}{\rho}\right)^n \sum_{z_1,\ldots,z_n=1}^{x-1}$ "吸收"到这些新的无标号图中，具体做法是让无标号图的每个顶点 z 单独贡献一个求和 $\frac{i}{\rho}\sum_{z=1}^{x-1}$ 。

Recall that the range of z in this sum ensures that the point z is always to the past of the points x or y in the causal set, since we are using a natural labelling for which all points not to past of x and y have larger labels than x (and hence also y). We can actually increase the range of this sum to $\sum_{z \in C}$, as in any diagram we know that the vertex z is connected via some route of directed edges to the vertices x or y , and hence there will be factors of G that are only non-trivial if the point z is to the past of the points x or y . Finally, the factors of $1/4\,!$ in (65) will cancel any numerical factors arising in the above procedure, up to the remaining symmetry factor of the unlabelled diagram. Henceforth, we call such an unlabelled diagram simply a diagram.

回顾可知，由于我们采用自然标号法，所有不在 x 和 y 过去的点的标号都大于 x (因此也大于 y), 故该求和中 z 的取值范围保证了在因果集里，点 z 始终位于点 x 或 y 的过去。实际上我们可以将该求和的范围扩大到 $\sum_{z \in C}$ ，因为在任意图中，顶点 z 都通过某条有向边路径与顶点 x 或 y 相连，因此会存在仅当点 z 位于点 x 或 y 过去时才非平凡的 G 因子。最后，式 (65) 中的 $1/4\,!$ 因子会抵消上述过程中产生的所有数值因子，仅剩下未标号图的剩余对称因子。此后，我们将这类图直接称为未标号图。

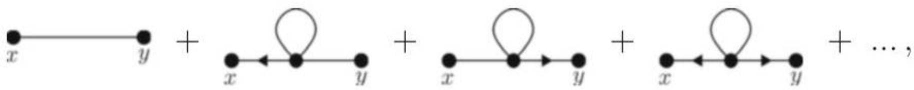
Summary of Analogue Feynman Diagrams and Rules

类比费曼图与规则总结

Following the previous section, we have

承接上一节内容，我们得到

$\langle\Omega|\,\phi_x^{(H)}\phi_y^{(H)}\,|\Omega\rangle =$



(66)

where the possible diagrams on the rhs consist of all the connected graphs one can draw by adding some number of internal vertices with four legs each (here we have only shown those with one internal vertex for brevity). One then "dresses" the graphs with directed edges in all possible ways such that (i) by following directed edges, every internal vertex has a route out to either x or y , or both, and (ii) every such route leads to x or y (thus, there are no closed loops of directed edges).

其中右侧的可能图包含所有可通过添加若干个各带四条边的内顶点得到的连通图 (为简便起见，此处仅展示含一个内顶点的图)。随后我们为图按所有可能方式赋予有向边“配置”:(i) 沿有向边行进，每个内顶点都存在通向 x 、或 y 、或同时通向二者的路径；(ii) 所有此类路径都终止于 x 或 y (因此不存在有向边构成的闭合回路)。

The contribution of a given diagram can be computed from the following rules:

给定图的贡献可通过以下规则计算:

1. For each internal vertex z , we get a sum $\frac{i}{\rho} \sum_{z \in C} \lambda_z$.

1. 对每个内顶点 z ， 我们得到求和项 $\frac{i}{\rho} \sum_{z \in C} \lambda_z$ 。

2. For each directed leg from a vertex a to b , we get a factor $iG_{ba} \cdot {}^{*} \cdot {}^{*}_{b}$

2. 对每条从顶点 a 到 b 的有向边，我们得到一个因子 $iG_{ba} \cdot {}^{*} \cdot {}^{*}_{b}$

3. For each undirected leg from a to b , we get a factor G^F_{ab} . * ⋮

3. 对每条从 a 到 b 的无向边，我们得到一个因子 G^F_{ab} 。* ⋮

4. For each internal vertex z , if it has p outgoing directed legs, we get a factor of $(-1)^{p-1}$.

4. 对每个内顶点 z ， 若它有 p 条出向有向边，我们得到一个因子 $(-1)^{p-1}$ 。

5. Divide by symmetry factor, S_D , of the diagram D . Equivalently, multiply by $1/S_D$.

5. 除以图 D 的对称因子 S_D 。等价地，乘以 $1/S_D$ 。

The symmetry factor, S_D , of a diagram D with n internal vertices, is simply the factor that is left over when we divide $(4!)^n$ by all the numerical factors we get in the construction of D following the steps of the previous section. It represents certain symmetries of the given diagram. For example, S_D contains a factor of 2 for any loops from a vertex to itself (e.g., the loop from z_2 to itself in Fig.3c), as the diagram is symmetric under the interchange of the ends of such a line. S_D contains a factor of q ! if the diagram is symmetric under the interchange of q lines (e.g., the two directed lines from z_1 to z_2 or the two undirected lines from z_1 to z_3 in Fig. 2e). Factors also arise if the diagram is symmetric under the interchange of certain vertices. In general, it can be non-trivial to determine the symmetry factor, so we will not go into this here.

含 n 个内顶点的图 D 的对称因子 S_D ， 就是我们遵循上一节步骤构造 D 时，用 $(4!)^n$ 除以所有得到的数值因子后剩余的因子。它代表给定图的特定对称性。例如，对任意顶点到自身的圈 (比如图 3c 中从 z_2 到自身的圈)，S_D 包含因子 2，因为交换此类边两端点后图保持对称。若交换 q 条边后图保持对称 (比如图 2e 中从 z_1 到 z_2 的两条有向边，或从 z_1 到 z_3 的两条无向边)，则 S_D 包含因子 q !。若交换特定顶点后图保持对称，也会产生额外因子。一般来说，确定对称因子并非易事，因此我们在此不展开讨论。

We now describe some notable features of our diagrams. Due to the construction process, all the diagrams are connected. In the continuum, one finds both connected and disconnected diagrams in the numerator and denominator of the rhs of (2) and further that only the connected diagrams remain after taking the quotient. In our case, no disconnected diagrams ever arise.

现在我们介绍这类图的几个显著特点。由于构造方式，所有图都是连通的。在连续场论中，(2) 式右侧的分子和分母既会出现连通图也会出现不连通图，取商后仅剩下连通图。在我们的情况中，永远不会产生不连通图。

Another interesting feature is that causality is explicitly encoded through the appearance of the retarded Green function. These factors ensure that the correlation between the fields at x and y only differs from the free correlation, $W_{xy} = \langle\Omega|\phi_x\phi_y|\Omega\rangle$, if either x or y sit to the future of at least one point z for which the interaction is turned on, i.e., $\lambda_z \neq 0$. This manifest causality in our diagrams is reminiscent of [15], though in that case they were concerned with sources and detectors. It would be interesting to see if further comparisons between our diagrams and those of [15] can be drawn.

另一个有趣的特点是，因果性通过推迟格林函数的出现被显式编码。这些因子保证了，只有当 x 或 y 至少有一个位于相互作用开启点 z 的未来 (即满足 $\lambda_z \neq 0$) 时，x 和 y 处场的关联才会不同于自由关联 $W_{xy} = \langle\Omega|\phi_x\phi_y|\Omega\rangle$。我们图中的这种显式因果性让人联想到文献 [15]，不过该文献研究的是源与探测器。进一步探究我们的图与文献 [15] 中的图能否做更多对比会是很有意义的工作。

Finally, the above procedure can be immediately generalized from the SJ state to any Gaussian state in the free theory for which n -point functions (not necessarily time-ordered) equal sums of products of 2-point functions. This latter feature is all that was really needed from the state in our prescription.

最后，上述过程可以直接从 SJ 态推广到自由理论中的任意高斯态，只要这类高斯态满足 n 点函数 (不必是时序的) 等于两点函数乘积的和。我们的构造方案中，仅要求态满足这后一个性质。

Discussion

讨论

We developed interacting real scalar QFT on a fixed causal set. We did this first via the double path integral framework, following Sorkin's suggestion in [3] to modify the analogue of the causal set action to include a self-interacting ϕ^4 term. We then used our interacting decoherence functional to derive the corresponding modification to the canonical theory. This amounted to a unitary transformation of the field operators at each causal set point, i.e., Equation (52). We further highlighted the similarities between the double path integral and canonical descriptions of our causal set interacting QFT and those of the continuum.

我们在固定因果集合上构建了相互作用实标量量子场论 (QFT)。我们首先通过双路径积分框架完成这一工作，遵循 Sorkin 在文献 [3] 中的建议，修改了因果集合作用量的类比形式，以引入自相互作用 ϕ^4 项。随后我们利用所得到的相互作用退相干泛函，推导出了正则理论对应的修正形式。这相当于对每个因果集合点处的场算子做幺正变换，即式 (52)。我们进一步强调了本文因果集合相互作用 QFT 的双路径积分描述、正则描述，与连续体情形中对应描述的相似性。

After answering some initial questions surrounding the interacting decoherence functional, we focused on the interacting 2-point function. We determined how to compute this 2-point function, order by order in the interaction parameter λ , using a diagrammatic approach. The diagrams that arose in our case resembled those of the continuum but with both directed and undirected edges. This construction can also be generalized to time-ordered or causally ordered n -point functions by including more factors of the field variables in the double path integral.

在回答了围绕相互作用退相干泛函的若干基础问题后，我们将重点放在了相互作用两点函数上。我们确定了如何利用图方法，按相互作用参数 λ 的阶逐阶计算该两点函数。我们得到的图类似连续体 QFT 的费曼图，但同时包含有向边与无向边。该构造也可以推广到时间序或因果序的 n 点函数，只需要在双路径积分中引入更多场变量因子即可。

With this basic framework of interacting QFT on causal sets laid out, there are many interesting directions for future investigations. One could compare the interacting 2-point function of the causal set with that of the continuum. For further comparison, one could use the above framework to determine scattering amplitudes between states of the free theory and compare them to the textbook continuum expressions.

在搭建好因果集合上相互作用 QFT 的基础框架后，未来仍有许多值得研究的有趣方向。研究者可以对比因果集合与连续体的相互作用两点函数。为进一步对比，还可以利用本文框架得到自由理论态之间的散射振幅，再和教科书中的连续体表达式做比较。

One important aspect of continuum interacting QFT, and one not considered here, is that of renormalization. To study this in the framework outlaid above, one would first need to specify some coarse-graining procedure, either by (i) removing some causal set points or by (ii) truncating the "higher-energy" modes from the eigenspectrum or by some other means altogether. With some coarse-graining procedure in place, one could then integrate out the appropriate degrees of freedom from the double path integral. For (i), one would simply integrate out the values of ξ_x and $\overline{\xi}_x$ for every point $x \in C$ that was thrown away. For (ii), one would first expand ξ and $\overline{\xi}$ in the eigenmodes of $i\Delta$ and then integrate out those modes above the cut-off. To study renormalization, one could then look at whether there is some dependence of the coupling λ on the cut-off that leaves the interacting n -point functions constant wrt the cut-off. Such a dependence could then be interpreted as the running of the coupling λ .

连续体相互作用 QFT 的一个重要方面 (也是本文未讨论的内容) 是重整化。要在上述框架中研究重整化，首先需要指定粗粒化流程，粗粒化可以通过 (i) 移除部分因果集合点、(ii) 从特征谱中截断“高能”模，或是其他任意方式实现。确定粗粒化流程后，就可以从双路径积分中积出对应自由度。对于方式 (i)，只需对每个被移除点 $x \in C$ ，积出 ξ_x 和 $\overline{\xi}_x$ 的取值即可。对于方式 (ii)，先将 ξ 和 $\overline{\xi}$ 按 $i\Delta$ 的本征模展开，再积出截断能量以上的模。要研究重整化，可以观察耦合常数 λ 是否对截断存在某种依赖关系，使得相互作用 n 点函数不随截断变化。这种依赖关系就可以被解释为耦合 λ 的跑动。

Cross-References

交叉引用

Entanglement Entropy and Causal Set Theory

纠缠熵与因果集理论

Quantum Field Theory on Causal Sets

因果集上的量子场论

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